Lecture Notes: Non-Standard Approach to J.F. Colombeau's Theory of Generalized Functions*

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Abstract

In these lecture notes we present an introduction to non-standard analysis especially written for the community of mathematicians, physicists and engineers who do research on J. F. Colombeau' theory of new generalized functions and its applications. The main purpose of our non-standard approach to Colombeau' theory is the improvement of the properties of the scalars of the varieties of spaces of generalized functions: in our non-standard approach the sets of scalars of the functional spaces always form algebraically closed non-archimedean Cantor complete fields. In contrast, the scalars of the functional spaces in Colombeau's theory are rings with zero divisors. The improvement of the scalars leads to other improvements and simplifications of Colombeau's theory such as reducing the number of quantifiers and possibilities for an axiomatization of the theory. Some of the algebras we construct in these notes have already counterparts in Colombeau's theory, other seems to be without counterpart. We present applications of the theory to PDE and mathematical physics. Although our approach is directed mostly to Colombeau's community, the readers who are already familiar with non-standard methods might also find a short and comfortable way to learn about Colombeau's theory: a new branch of functional analysis which naturally generalizes the Schwartz

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theory of distributions with numerous applications to partial differential equations, differential geometry, relativity theory and other areas of mathematics and physics.

MSC: Functional Analysis (46F30); Generalized Solutions of PDE (35D05).

1 Introduction

This lecture notes are an extended version of the several lectures I gave at the University of Vienna during my visit in the Spring of 2006. My audience consisted mostly of colleagues, graduate and undergraduate students who do research on J.F. Colombeau's non-linear theory of generalized functions (J.F. Colombeau's ([10]-[15]) and its applications to ordinary and partial differential equations, differential geometry, relativity theory and mathematical physics. With very few exceptions the colleagues attended my talks were not familiar with nonstandard analysis. This fact strongly influenced the nature of my lectures and these lecture notes. I do not assume that the reader of these notes is necessarily familiar neither with A. Robonson's non-standard analysis (A. Robonson [75]) nor with A. Robonson's non-standard asymptotic analysis (A. Robinson [76] and A. Robonson and A.H. Lightstone [57]). I have tried to downplay the role of mathematical logic as much as possible. With examples from Colombeau's theory I tried to convince my colleagues that the involvement of the non-standard methods in Colombeau theory has at least the following three advantages:

1. The scalars of the non-standard version of Colombeau's theory are algebraically closed Cantor complete fields. Recall that in Colmbeau's theory the scalars of the functional spaces are rings with zero divisors.

2. The involvement of non-standard analysis in Colombeau's theory leads to simplification of the theory by reducing the number of the quantifiers. This should be not of surprise because non-standard analysis is famous with the so called reduction of quantifiers. For comparison, the familiar definition of a limit of a function in standard analysis involves three (non-commuting) quantifiers. In contrast, its non-standard characterization uses only one quantifier. Another example gives the definition of a compact set in point set topology involves at least two quantifiers. In contrast, there is a free of quantifiers non-standard characterization of the compactness in terms of monads. Since Colombeau' theory is relatively heavy of quantifiers, the reduction of the numbers of quantifiers makes the theory more attractive to colleagues outside the Colombeau's community and in particular to theoretical physicists.

3. In my lectures and in these notes I decided to follow mostly the so called constructive version of the non-standard analysis where the non-

standard real number $a \in {}^*\mathbb{R}$ is equivalence class of families (a_i) in the ultrapower $\mathbb{R}^{\mathcal{I}}$ for some infinite set $\mathcal{I}$. Similarly, every non-standard smooth function $f \in {}^*\mathcal{E}(\Omega)$ is defined as equivalence class of families (f_i) in the ultrapower $\mathcal{E}(\Omega)^{\mathcal{I}}$. Here $\mathcal{E}(\Omega)$ is a (short) notation for $\mathcal{C}^\infty(\Omega)$. The equivalence relation in both $\mathbb{R}^{\mathcal{I}}$ and $\mathcal{E}(\Omega)^{\mathcal{I}}$ is defined in terms of a free ultrafilter $\mathcal{U}$ on $\mathcal{I}$. In our approach the choice of the index set $\mathcal{I}$ and the choice of the ultrafilter $\mathcal{U}$ are borrowed from Colombeau's theory. This approach to non-standard analysis is more directly connected with the standard (real) analysis and allow to involve the non-standard analysis in research with comparatively limited knowledge in the non-standard theory. The non-standard analysis however has also axiomatic version based on two axioms known a Saturation Principle and Transfer Principle. The involvement of non-standard analysis, if based on these two principles, opens the opportunities for axiomatization of Colombeau's theory. I have demonstrated this in the notes by presenting a couple of proofs to several theorems: one using families (nets), and another using these two axioms. The first might be more convincible for beginners to non-standard analysis but the second proofs are more elegant and short because it does not involve the representatives of the generalized numbers and generalized functions.

Let $\mathcal{T}$ stand for the usual topology on $\mathbb{R}^d$. J.F. Colombeau's non-linear theory of generalized functions is based on varieties of families of differential commutative rings $\mathcal{G} \stackrel{\text{def}}{=} \{\mathcal{G}(\Omega)\}_{\Omega \in \mathcal{T}}$ such that: 1) Each $\mathcal{G}$ is a **sheaf** of differential rings (consequently, each $f \in \mathcal{G}(\Omega)$ has a **support** which is a closed set of Ω). 2) Each $\mathcal{G}(\Omega)$ is supplied with a chain of sheaf-preserving embeddings $\mathcal{C}^\infty(\Omega) \subset \mathcal{D}'(\Omega) \subset \mathcal{G}(\Omega)$, where $\mathcal{C}^\infty(\Omega)$ is a **differential subring** of $\mathcal{G}(\Omega)$ and the space of L. Schwartz's distributions $\mathcal{D}'(\Omega)$ is a **differential linear subspace** of $\mathcal{G}(\Omega)$. 3) The ring of the scalars $\widetilde{\mathbb{C}}$ of the family $\mathcal{G}$ (defined as the set of the functions in $\mathcal{G}(\mathbb{R}^d)$ with zero gradient) is a non-Archimedean ring with zero devisors containing a copy of the complex numbers $\mathbb{C}$. Colombeau theory has numerous applications to ordinary and partial differential equations, fluid mechanics, elasticity theory, quantum field theory and more recently to general relativity.

2 κ-Good Two Valued Measures

I follow the philosophy that every non-standard real number $a \in {}^*\mathbb{R}$ is, roughly speaking, a family (a_i) in the ultrapower $\mathbb{R}^{\mathcal{I}}$ for some infinite set $\mathcal{I}$. Similarly, every nonstandard smooth function $f \in {}^*\mathcal{E}(\Omega)$ is again, roughly speaking, a family (f_i) in the ultrapower $\mathcal{E}(\Omega)^{\mathcal{I}}$. Here $\mathcal{E}(\Omega)$ is a (short) notation for $\mathcal{C}^\infty(\Omega)$.

2.1 Definition (κ-Good Two Valued Measures)**.** Let $\mathcal{I}$ be an infinite set of cardinality κ, i.e. $\mathrm{card}(\mathcal{I}) = \kappa$. A mapping $p : \mathcal{P}(\mathcal{I}) \to \{0, 1\}$ is a κ-**good two-valued (probability) measure** if

1. p is finitely additive, i.e. $p(A \cup B) = p(A) + p(B)$ for disjoint A and B.

2. $p(\mathcal{I}) = 1$.

3. $p(A) = 0$ for finite A.

4. There exists a sequence of sets $(\mathcal{I}_n)$ such that

   (a) **(a)** $\mathcal{I} \supset \mathcal{I}_1 \supset \mathcal{I}_2 \supset \dots$,
   (b) **(b)** $\mathcal{I}_n \setminus \mathcal{I}_{n-1} \neq \varnothing$ for all n,
   (c) **(c)** $\bigcap_{n=1}^\infty \mathcal{I}_n = \varnothing$,
   (d) **(d)** $p(\mathcal{I}_n) = 1$ for all n.

5. If $\mathcal{I}$ is uncountable, we impose one more property: p should be κ-**good** in the sense that for every set $\Gamma \subseteq \mathcal{I}$, with $\mathrm{card}(\Gamma) \leq \kappa$, and every **reversal** $R : \mathcal{P}_\omega(\Gamma) \to \mathcal{U}$ there exists a **strict reversal** $S : \mathcal{P}_\omega(\Gamma) \to \mathcal{U}$ such that $S(X) \subseteq R(X)$ for all $X \in \mathcal{P}_\omega(\Gamma)$. Here $\mathcal{P}_\omega(\Gamma)$ denotes the set of all finite subsets of Γ and $\mathcal{U} = \{A \in \mathcal{P}(\mathcal{I}) \mid P(A) = 1\}$.

2.2 Remark (Reversals)**.** Let $\Gamma \subseteq I$. A function $R : \mathcal{P}_\omega(\Gamma) \to \mathcal{U}$ is called a **reversal** if $X \subseteq Y$ implies $R(X) \supseteq R(Y)$ for every $X, Y \in \mathcal{P}_\omega(\Gamma)$. A function $S : \mathcal{P}_\omega(\Gamma) \to \mathcal{U}$ is called a **strict reversal** if $S(X \cup Y) = S(X) \cap S(Y)$ for every $X, Y \in \mathcal{P}_\omega(\Gamma)$. It is clear that every strict reversal is a reversal (which justifies the terminology).

3 Existence of Two Valued κ-Good Measures

3.1 Theorem (Existence of Two Valued κ-Good Measures). *Let $\mathcal{I}$ be an infinite set and let $(\mathcal{I}_n)$ be a sequence of sets with the properties (a)-(c) (think of Colombeau's theory). Then there exists a two valued κ-good measure $p : \mathcal{P}(\mathcal{I}) \to \{0,1\}$, where $\kappa = \mathrm{card}(\mathcal{I})$, such that $p(\mathcal{I}_n) = 1$ for all $n \in \mathbb{N}$.*

3.2 Remark. We should note that for every infinite set $\mathcal{I}$ there exists a sequence $(\mathcal{I}_n)$ with the properties (a)-(c).

Proof: *Step 1:* **Define $\mathcal{F}_0 \subset \mathcal{P}(\mathcal{I})$ by**

$$\mathcal{F}_0 = \{A \in \mathcal{P}(\mathcal{I}) \mid \mathcal{I}_n \subseteq A \text{ for some } n\}.$$

It is easy to check that $\mathcal{F}_0$ **is a free countably incomplete filter on** $\mathcal{I}$ in the sense that $\mathcal{F}_0$ has the following properties:

1. $\varnothing \notin \mathcal{F}_0$.

2. $\mathcal{F}_0$ is closed under finite intersections.

3. $\mathcal{F}_0 \ni A \subseteq B \in \mathcal{P}(\mathcal{I})$ implies $B \in \mathcal{F}_0$.

4. $\mathcal{I}_n \in \mathcal{F}_0$ for all $n \in \mathbb{N}$.

   Step 2: **We extend $\mathcal{F}_0$ to a ultrafilter $\mathcal{U}$ on $\mathcal{I}$ by Zorn lemma:** Let $\mathcal{L}$ denote the set of all free filter $\mathcal{F}$ on $\mathcal{I}$ containing $\mathcal{I}_n$, i.e.

$$\mathcal{L} = \{\mathcal{F} \subset \mathcal{P}(\mathcal{I}) \mid \mathcal{F} \text{ satisfies (i)-(iv), where } \mathcal{F}_0 \text{ should be replaced by } \mathcal{F}\}.$$

We shall order $\mathcal{L}$ by inclusion $\subset$. Observe that every chain L in $\mathcal{L}$ is bounded from above by $\bigcup_{A \in L} A$ and it is not difficult to show that $\bigcup_{A \in L} A \in \mathcal{L}$. Thus $\mathcal{L}$ has maximal elements $\mathcal{U}$ by Zorn lemma. In what follows we shall keep $\mathcal{U}$ fixed.

   Step 3: We shall prove now that $\mathcal{U}$ **has the following (free ultrafilter) properties:**

1. $\varnothing \notin \mathcal{U}$.

2. $\mathcal{U}$ is closed under finite intersections.

3. $\mathcal{U} \ni A \subseteq B \in \mathcal{P}(\mathcal{I})$ implies $B \in \mathcal{U}$.

4. **(4)** $\mathcal{I}_n \in \mathcal{U}$ for all $n \in \mathbb{N}$.

5. **(5)** $A \cup B \in \mathcal{U}$ implies either $A \in \mathcal{U}$ or $B \in \mathcal{U}$.

Indeed, $\mathcal{U}$ satisfies (1)-(4) by the choice of $\mathcal{U}$ since $\mathcal{U} \in \mathcal{L}$. To show the property (5), suppose (on the contrary) that $A \cup B \in \mathcal{U}$ and $A, B \notin \mathcal{U}$ for some subsets A and B of $\mathcal{I}$. Next, we observe that $\mathcal{F}_A = \{X \in \mathcal{P}(\mathcal{I}) \mid A \cup X \in \mathcal{U}\}$ is also a free filter on $\mathcal{I}$ (i.e. $\mathcal{F}_A$ satisfies the properties (1)-(4)). Next, we observe that $\mathcal{F}_A$ is a proper extension of $\mathcal{U}$ since $B \in \mathcal{F}_A \setminus \mathcal{U}$ by the assumption for B, contradicting the maximality of $\mathcal{U}$.

Step 4: **Define** $p : \mathcal{P}(\mathcal{I}) \to \{0, 1\}$ by $p(A) = 1$ whenever $A \in \mathcal{U}$ and $p(A) = 0$ whenever $A \notin \mathcal{U}$. We have to show now that p is a κ-good two valued measure (Definition 2.1). To check the finite additivity property (i) of p, suppose that $A \cap B = \varnothing$ for some $A, B \in \mathcal{P}(\mathcal{I})$. Suppose, first, that $A \cup B \in \mathcal{U}$, so we have $p(A \cup B) = 1$. On the other hand, by properties (1) and (5), exactly one of the following two statements is true: either (a) $A \in \mathcal{U}$ and $A \notin \mathcal{U}$ or (b) $A \notin \mathcal{U}$ and $A \in \mathcal{U}$. In either case we have $p(A) + p(B) = 1$, as required. Suppose, now, that $A \cup B \notin \mathcal{U}$, so we have $p(A \cup B) = 0$. In this case we have $A \notin \mathcal{U}$ and $B \notin \mathcal{U}$ by property (3). Thus $p(A) + p(B) = 0$. The property (ii): $p(\mathcal{I}) = 1$ holds since $\mathcal{I} \in \mathcal{U}$ by properties (3) and (4) of $\mathcal{U}$. To prove the property (iii), suppose (on the contrary) that $p(A) = 1$ for some finite set $A \subset \mathcal{I}$, i.e. $A \in \mathcal{U}$. It follows that there exists $i \in A$ such that $\{i\} \in \mathcal{U}$ by property (5) of $\mathcal{U}$ since we have $\bigcup_{i \in A}\{i\} = A$. Thus $\{i\} \in \mathcal{I}_n$ for all $n \in \mathcal{N}$ by properties (1), (2) and (4) of $\mathcal{U}$. It follows that $\{i\} \in \bigcap_{n \in \mathbb{N}} \mathcal{I}_n$ contradicting property (c) of the sequence $(\mathcal{I}_n)$. The property (iv) holds by the choice of $\mathcal{U}$ since $\mathcal{I}_n \in \mathcal{F}_0 \subset \mathcal{U}$ thus $p(\mathcal{I}_n) = 1$. For the proof of the property (v) of the measure p we shall refer to C. C. Chang and H. J. Keisler [8] or to T. Lindstrøm [56]. ▲

4 A Non-Standard Analysis: The General Theory

4.1 Definition (A Non-Standard Extension of a Set)**.** Let S be a set and $\mathcal{I}$ be and infinite set, and $S^{\mathcal{I}}$ be the corresponding ultrapower.

1. We say that (a_i) and (b_i) are equal almost everywhere in $\mathcal{I}$, in symbol $a_i = b_i$ a.e., if $p(\{i \in \mathcal{I} \mid a_i = b_i \text{ in } S\}) = 1$, or equivalently, if $\{i \in \mathcal{I} \mid a_i = b_i \text{ in } S\} \in \mathcal{U}$, where $\mathcal{U} = \{A \in \mathcal{P}(\mathcal{I}) \mid p(A) = 1\}$. We

denote by $\sim$ the corresponding equivalence relation, i.e. $(a_i) \sim (b_i)$ if $a_i = b_i$ a.e..

2. We denote by $\langle a_i \rangle$ the equivalence class determined by (a_i). The set of all equivalence classes $^*S = S^{\mathcal{I}}/_{\sim}$ is called a **non-standard extension** of S.

3. Let $s \in S$. We define $^*s = \langle a_i \rangle$, where $a_i = s$ for all $i \in \mathcal{I}$. We define the canonical embedding $\sigma : S \to {}^*S$ by $\sigma(s) = {}^*s$, and denote by $^{\sigma}S = \{^*s \mid s \in S\}$ the range of σ. We shall sometimes treat this embedding as an inclusion, $S \subseteq {}^*S$, by letting $s = {}^*s$ for all $s \in S$.

4. More generally, if $X \subseteq S$, we define $^*X \subseteq {}^*S$ by

$$^*X = \{\langle x_i \rangle \in {}^*S \mid x_i \in X \text{ a.e.}\}.$$

We have $X \subseteq {}^*X$ under the embedding $x \to {}^*x$. We say that $^*X$ is the **non-standard extension** of X.

4.2 Theorem (Axiom 1. Extension Principle). *Let S be a set. Then $S \subseteq {}^*S$ and $S = {}^*S$ iff S is a finite set.*

Proof. $S \subseteq {}^*S$ holds in the sense of the embedding σ. Suppose, first, that S is a finite set and let $\langle a_i \rangle \in {}^*S$. We observe that the finite collection of sets $\{i \in \mathcal{I} \mid a_i = s\}$, $s \in S$, are mutually disjoint and $\bigcup_{s \in S}\{i \in \mathcal{I} \mid a_i = s\} = \mathcal{I}$. Thus $\sum_{s \in S} p(\{i \in \mathcal{I} \mid a_i = s\}) = 1$ by the finite additivity of the measure p. It follows that there exists a unique $s_0 \in S$ such that $p(\{i \in \mathcal{I} \mid a_i = s_0\}) = 1$ (and $p(\{i \in \mathcal{I} \mid a_i = s_0\}) = 0$ for all $s \in S, s \neq s_0$). Thus we have $\langle a_i \rangle = {}^*s_0 \in S$, as required. Suppose now, that S is an infinite set. We have to show that $^*S \setminus S \neq \varnothing$. Indeed, by axiom of choice, there exists a sequence (s_n) in S such that $s_m \neq s_n$ whenever $m \neq n$. Next, we define $(a_i) \in S^{\mathcal{I}}$ by $a_i = s_n$, where $n = \max\{m \in \mathbb{N} \mid i \in \mathcal{I}_{m-1} \setminus \mathcal{I}_m\}$ and we have let also $\mathcal{I}_0 = \mathcal{I}$. Let $s \in S$. We have to show that the set $\{i \in \mathcal{I} \mid a_i \neq s\}$ is of measure 1. Indeed, if s is not in the range of (s_n), then $\{i \in \mathcal{I} \mid a_i \neq s\} = \mathcal{I}$ and is of measure 1. If s is in the range of (s_n), then $s = s_k$ for exactly one $k \in \mathbb{N}$. We observe that $\mathcal{I}_k \subseteq \{i \in \mathcal{I} \mid a_i \neq s\}$. Now the set $\{i \in \mathcal{I} \mid a_i \neq s\}$ is of measure 1 because $\mathcal{I}_k$ is of measure one, by property (iv)-(c) of p. The proof is complete. Thus $\langle s_i \rangle \in {}^*S \setminus S$ as required. $\square$

In what follows $(A_i) \in \mathcal{P}(S)^{\mathcal{I}}$ means that $A_i \subseteq S$ for all $i \in \mathcal{I}$.

4.3 Definition (Internal Sets)**.** Let $\mathcal{A} \subseteq {}^*S$. We say that $\mathcal{A}$ is an **internal set** of ${}^*S$ if there exists a family $(A_i) \in \mathcal{P}(S)^{\mathcal{I}}$ of subsets of S such that

$$\mathcal{A} = \{\langle s_i \rangle \in {}^*S \mid s_i \in A_i \ \text{ a.e. }\}.$$

We say that the family (A_i) generates $\mathcal{A}$ and we write $\mathcal{A} = \langle A_i \rangle$. Let, in the particular, $A_i = A$ for all $i \in \mathcal{I}$ and some $A \subseteq S$. We say that the internal set ${}^*A = \langle A_i \rangle$ is the **non-standard extension** of A. We denote by ${}^*\mathcal{P}(S)$ the set of the internal subsets of ${}^*S$. The sets in ${}^*\mathcal{P}(S) \setminus \mathcal{P}(S)$ are call **external**.

If $X \subseteq S$, then ${}^*X$ is internal and ${}^*X$ is generated by the constant family $X_i = X$ for all $i \in \mathcal{I}$. In particular ${}^*S$ is an internal set. Let $\langle s_i \rangle \in {}^*S \setminus S$ be the element defined in the proof of Theorem 4.2. Then the singleton $\{\langle s_i \rangle\}$ is an internal set which is not of the form ${}^*X$ for some $X \subseteq S$. This internal set is generated by the singletons $\{s_i\}$, i.e. $\{\langle s_i \rangle\} = \langle \{s_i\} \rangle$. More generally, every finite subset of ${}^*S$ is an internal set. We shall give more examples of infinite internal sets of ${}^*\mathbb{R}$ and ${}^*\mathbb{C}$ in the next section. If $A \subseteq S$, then A is an external set of ${}^*S$.

In the next theorem we use for the first time the property (v) of the probability measure p (Definition 2.1). Recall that $\kappa = \mathrm{card}(\mathcal{I})$."

4.4 Theorem (Axiom 2. Saturation Principle in ${}^*\mathbb{C}$)**.** *${}^*\mathbb{C}$ is κ-saturated in the sense that every family $\{\mathcal{A}_\gamma\}_{\gamma \in \Gamma}$ of internal sets of ${}^*\mathbb{C}$ with the finite intersection property, and an index set Γ with $\mathrm{card}(\Gamma) \leq \kappa$, has a non-empty intersection.*

Proof: We have (by assumption) that $\bigcap_{\gamma \in F} \mathcal{A}_\gamma \neq \varnothing$ for every finite subset F of Γ. We have to show that $\bigcap_{\gamma \in \Gamma} \mathcal{A}_\gamma \neq \varnothing$. The fact that $\mathcal{A}_\gamma$ is an internal set means that $\mathcal{A}_\gamma = \langle \mathbb{A}_{\gamma,i} \rangle$ for some $\mathbb{A}_{\gamma,i} \subseteq \mathbb{C}$. Hence, for every finite subset F of Γ we have $\{i \in \mathcal{I} : \bigcap_{\gamma \in F} \mathbb{A}_{\gamma,i} \neq \varnothing\} \in \mathcal{U}$. Next, we define the function $R : \mathcal{P}_\omega(\Gamma) \to \mathcal{U}$, by

$$R(F) = \mathcal{I}_{\mathrm{card}(F)} \cap \{i \in \mathcal{I} : \bigcap_{\gamma \in F} \mathbb{A}_{\gamma,i} \neq \varnothing\},$$

for every finite subset F of Γ. It is clear that R is a reversal (Remark 2.2). Since p is a κ-good measure, it follows that there exists a strict reversal $S : \mathcal{P}_\omega(\Gamma) \to \mathcal{U}$ which minorizes R, i.e.

$$S(F) \subseteq \mathcal{I}_{\mathrm{card}(F)} \cap \{i \in \mathcal{I} : \bigcap_{\gamma \in F} \mathbb{A}_{\gamma,i} \neq \varnothing\},$$

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for every finite subset F of Γ. For every $i \in \mathcal{I}$ we define

$$\Gamma_i = \{\gamma \in \Gamma \mid i \in S(\{\gamma\})\}.$$

Notice that if $\operatorname{card}(\Gamma_i) = m$ for some $m \in \mathbb{N}$ and some $i \in \mathcal{I}$, then $i \in \mathcal{I}_m$. Indeed, $\operatorname{card}(\Gamma_i) = m$ means that $\Gamma_i = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ for some distinct $\gamma_1, \gamma_2, \ldots, \gamma_m \in \Gamma$ such that $i \in \bigcap_{n=1}^m S(\gamma_n)$. Using the fact that S is a strict reversal, we have $\bigcap_{n=1}^m S(\gamma_n) = S(\{\gamma_1, \gamma_2, \ldots, \gamma_m\}) \subseteq R(\{\gamma_1, \gamma_2, \ldots, \gamma_m\}) \subseteq \mathcal{I}_m$, hence, $i \in \mathcal{I}_m$ follows. On the other hand, $\bigcap_{m=1}^\infty \mathcal{I}_m = \varnothing$ implies that Γ_i is a finite set for every $i \in \mathcal{I}$. As a result, $\bigcap_{\gamma \in \Gamma_i} \mathbb{A}_{\gamma,i} \neq \varnothing$ for all $i \in \mathcal{I}$. By Axiom of Choice, there exists $(A_i) \in \mathbb{C}^{\mathcal{I}}$ such that $A_i \in \bigcap_{\gamma \in \Gamma_i} \mathbb{A}_{\gamma,i}$ for all $i \in \mathcal{I}$. We intend to show that $\langle A_i \rangle \in \bigcap_{\gamma \in \Gamma} \mathcal{A}_\gamma$. Indeed, for every $\gamma \in \Gamma$ we have

$$S(\{\gamma\}) = \{i \mid \gamma \in \Gamma_i\} \subseteq \{i \mid A_i \in \mathbb{A}_{\gamma,i}\}.$$

Since $S(\{\gamma\}) \in \mathcal{U}$, it follows that $\{i \in \mathcal{I} \mid A_i \in \mathbb{A}_{\gamma,i}\} \in \mathcal{U}$. Hence $\langle A_i \rangle \in \langle \mathbb{A}_{\gamma,i} \rangle = \mathcal{A}_\gamma$, as required.
▲

In what follows we use the notation $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

4.5 Theorem (Sequential Saturation). *$^*S$ is sequentially saturated in the sense that every sequence $\{\mathcal{A}_n\}_{n \in \mathbb{N}_0}$ of internal sets of $^*S$ with the finite intersection property has a non-empty intersection.*

Proof 1 (An Indirect Proof): An immediate consequence of Theorem 4.4 in the case of countable index set Γ.
Proof 2 (A Direct Proof): We have $\bigcap_{n=0}^m \mathcal{A}_n \neq \varnothing$ for all $m \in \mathbb{N}_0$, by assumption. We have to show that $\bigcap_{n=0}^\infty \mathcal{A}_n \neq \varnothing$. The fact that $\mathcal{A}_n$ are internal sets means that $\mathcal{A}_n = \langle \mathbb{A}_{n,i} \rangle$ for some $\mathbb{A}_{n,i} \subseteq \mathbb{C}$, where $n \in \mathbb{N}_0$, $i \in \mathcal{I}$. We have $\langle \bigcap_{n=0}^m \mathbb{A}_{n,i} \rangle = \bigcap_{n=0}^m \langle \mathbb{A}_{n,i} \rangle = \bigcap_{n=0}^m \mathcal{A}_n \neq \varnothing$. Thus for every $m \in \mathbb{N}_0$ we have

$$(1) \qquad \Phi_m = \{i \in \mathcal{I} \mid \cap_{n=0}^m \mathbb{A}_{n,i} \neq \varnothing\} \in \mathcal{U}.$$

Without loss of generality we can assume that $\mathbb{A}_{0,i} \neq \varnothing$ for all $i \in \mathcal{I}$ (indeed, if $\Phi_0 \neq \mathcal{I}$, we can choose another representative of $\mathcal{A}_0$ by $\mathbb{A}'_{0,i} = \mathbb{A}_{0,i}$ for $i \in \Phi_0$ and by $\mathbb{A}'_{0,i} = \mathbb{C}$ for $i \in \mathcal{I} \setminus \Phi_0$). Next, we define the function $\mu : \mathcal{I} \to \mathbb{N}_0 \cup \{\infty\}$, by

$$\mu(i) = \max\{m \in \mathbb{N}_0 \mid \cap_{n=0}^m \mathbb{A}_{n,i} \neq \varnothing\}.$$

Notice that μ is well-defined because the set

$$\{m \in \mathbb{N}_0 \mid \cap_{n=0}^{m} \mathbb{A}_{n,i} \neq \varnothing\},$$

is non-empty for all $i \in \mathcal{I}$ due to our assumption for $\mathbb{A}_{0,i}$. Thus we have $\bigcap_{n=0}^{\mu(i)} \mathbb{A}_{n,i} \neq \varnothing$ for all $i \in \mathcal{I}$. Hence (by Axiom of Choice) there exists $(A_i) \in \mathbb{C}^{\mathcal{I}}$ such that $A_i \in \bigcap_{n=0}^{\mu(i)} \mathbb{A}_{n,i}$ for all $i \in \mathcal{I}$. We intend to show that $\langle A_i \rangle \in \bigcap_{n=0}^{\infty} \mathcal{A}_n$ or equivalently, to show that for every $m \in \mathbb{N}_0$ we have $\{i \in \mathcal{I} \mid A_i \in \mathbb{A}_{m,i}\} \in \mathcal{U}$. We observe that

$$\Phi_m \subseteq \{i \in \mathcal{I} \mid A_i \in \mathbb{A}_{m,i}\}.$$

Indeed, $i \in \Phi_m$ implies $\bigcap_{n=0}^{m} \mathbb{A}_{n,i} \neq \varnothing$ which implies $0 \leq m \leq \mu(i)$ (by the definition of $\mu(i)$) leading to $A_i \in \mathbb{A}_{m,i}$, by the choice of (A_i). On the other hand, we have $\Phi_m \in \mathcal{U}$, by (1) implying $\{i \in \mathcal{I} \mid A_i \in \mathbb{A}_{m,i}\} \in \mathcal{U}$, as required, by property (3) of $\mathcal{U}$. ▲

4.6 Definition (Superstructure). *Let S be an infinite set. The **superstructure** $V(S)$ on S is the union*

$$V(S) = \bigcup_{n=0}^{\infty} V_n(S),$$

where the $V_n(S)$ are defined inductively by

$$V_0(S) = S, \quad V_1(S) = S \cup \mathcal{P}(S),$$
(2)
$$V_{n+1}(S) = V_n(S) \cup \mathcal{P}(V_n(S)).$$

*The members of $V(S)$ are called **entities**. The members of $V(S) \setminus S$ are called the **sets** of the superstructure $V(S)$ and the members of S are called the **individuals** of the superstructure $V(S)$.*

4.7 Definition (The Language $\mathcal{L}(V(S))$). *The language $\mathcal{L}(V(S))$ is the usual "language of the analysis" with the following restrictions: All quantifiers are bounded by sets in the superstrucure $V(S)$, i.e. quantifiers appear in the formulae of the language $\mathcal{L}(V(S))$ only in the form*

$$(\forall x \in A)P(x) \quad or \quad (\exists x \in A)P(x),$$

where $P(x)$ is a predicate in one or more variables and $A \in V(S) \setminus S$. In particular, formulae such as

$$(\forall x)P(x),$$
$$(\exists x)P(x),$$
$$(\forall x \in s)P(x),$$
$$(\exists x \in s)P(x),$$

where $s \in S$, do not belong the the language $\mathcal{L}(V(S))$.

In what follow $V(^{*}S)$ stands for the supersructure of $^{*}S$ and $\mathcal{L}(V(^{*}S))$ stands for the language on $V(^{*}S)$ which are defined exactly as $V(S)$ and $\mathcal{L}(V(S))$ after replacing S by $^{*}S$.

4.8 Theorem (Axiom 3. Transfer Principle)**.** *Let $P(x_1, x_2, \ldots x_n)$ be a predicate in $\mathcal{L}(V(S))$ and $A_1, A_2, \ldots, A_n \in V(S)$. Then $P(A_1, A_2, \ldots A_n)$ is true $\mathcal{L}(V(S))$ iff $P(^{*}A_1, {}^{*}A_2, \ldots {}^{*}A_n)$ is true in $\mathcal{L}(V(^{*}S))$.*

For examples of application of the Transfer Principle we refer to the first proofs of Lemma 5.3 and Lemma 5.4 later in this text.

5 A. Robinson's Non-Standard Numbers

In this section we apply the non-standard construction in the particular case $S = \mathbb{C}$, where $\mathbb{C}$ is the field of the complex numbers.

5.1 Definition (Non-Standard Numbers)**.**   *1. We define the* **complex non-standard numbers** *as the factor ring $^{*}\mathbb{C} = \mathbb{C}^{\mathcal{I}}/\sim$, where $(a_i) \sim (b_i)$ if $a_i = b_i$ a.e., i.e. if*

$$p\left(\{i \in \mathcal{I} \mid a_i = b_i\}\right) = 1$$

(or, equivalently, if $\{i \in \mathcal{I} \mid a_i = b_i\} \in \mathcal{U}$, where $\mathcal{U} = \{A \in \mathcal{P} \mid p(A) = 1\}$.) We denote by $\langle a_i \rangle \in {}^{*}\mathbb{C}$ the equivalence class determined by (a_i). The algebraic operations and the absolute value in $^{*}\mathbb{C}$ is inherited from $\mathbb{C}$. For example, $|\langle x_i \rangle| = \langle |x_i| \rangle$.

    2. The set of **real non-standard numbers** $^{*}\mathbb{R}$ *is (by definition) the non-standard extension of $\mathbb{R}$, i.e.*

$$^{*}\mathbb{R} = \{\langle x_i \rangle \in {}^{*}\mathbb{C} \mid x_i \in \mathbb{R} \quad a.e. \}.$$

The order relation if $^*\mathbb{R}$ is defined by $\langle a_i \rangle < \langle b_i \rangle$ if $a_i < b_i$ in $\mathbb{R}$ a.e., i.e. if

$$p\left(\{i \in \mathcal{I} \mid a_i < b_i\}\right) = 1.$$

3. *The mapping $r \to {}^*r$ defines an embeddings $\mathbb{C} \subset {}^*\mathbb{C}$ and $\mathbb{R} \subset {}^*\mathbb{R}$ by the constant nets, i.e. ${}^*r = \langle a_i \rangle$, where $a_i = r$ for all $i \in \mathcal{I}$.*

5.2 Theorem (Basic Properties).   1. *$^*\mathbb{C}$ is an algebraically closed non-archimedean field.*

2. *$^*\mathbb{R}$ is a real closed (totally ordered) non-archimedean field.*

Proof. We shall separate the proof of the above theorem in several small lemmas and prove some of them. We shall present also two proofs to each of the lemmas; one of them based on the Saturation Principle (Theorem 4.8) and the other on the properties of the measure p. The content of the next lemma is a small (but typical) part of the statement that both $^*\mathbb{C}$ and $^*\mathbb{R}$ are fields   □

5.3 Lemma (No Zero Divisors). *$^*\mathbb{C}$ is free of zero divisors.*

Proof 1: The statement

$$(\forall x, y \in \mathbb{C})(xy = 0 \Rightarrow x = 0 \vee y = 0),$$

is true because $\mathbb{C}$ is free of zero divisors. Thus

$$(\forall x, y \in {}^*\mathbb{C})(xy = 0 \Rightarrow x = 0 \vee y = 0),$$

is true (as required) by Transfer Principle (Theorem 4.8).
  ▲

Proof 2: Suppose $\langle a_i \rangle \langle b_i \rangle = 0$ in $^*\mathbb{C}$ for some $\langle a_i \rangle$, $\langle b_i \rangle \in {}^*\mathbb{C}$. Thus $\langle a_i b_i \rangle = 0$ implying $p(\{i \in \mathcal{I} \mid a_i b_i = 0\}) = 1$. On the other hand,

$$\{i \in \mathcal{I} \mid a_i b_i = 0\} = \{i \in \mathcal{I} \mid a_i = 0\} \cup \{i \in \mathcal{I} \mid b_i = 0\},$$

because $\mathbb{C}$ is free of zero divisors. It follows that

$$p(\{i \in \mathcal{I} \mid a_i = 0\}) + p(\{i \in \mathcal{I} \mid b_i = 0\}) \geq 1,$$

by the additivity of p. Since the range of p is $\{0, 1\}$, it follows that ether $p(\{i \in \mathcal{I} \mid a_i = 0\}) = 1$ or $p(\{i \in \mathcal{I} \mid b_i = 0\}) = 1$, i.e. either $\langle a_i \rangle = 0$ or $\langle b_i \rangle = 0$, as required. ▲

5.4 Lemma (Trichotomy). *Let $a, b \in {}^*\mathbb{R}$. Then ether $a < b$ or $a = b$ or $a > b$.*

Proof 1: The statement

$$(\forall x, y \in \mathbb{R})(x \neq y \Rightarrow x < y \lor x > y),$$

is true because $\mathbb{R}$ is a totally ordered set. Thus

$$(\forall x, y \in {}^*\mathbb{R})(x \neq y \Rightarrow x < y \lor x > y),$$

is true (as required) by Transfer Principle (Theorem 4.8).
▲

Proof 2: Suppose that $\langle a_i \rangle, \langle b_i \rangle \in {}^*\mathbb{R}$. We observe that the sets

$$A = \{i \in \mathcal{I} \mid a_i < b_i\}, \quad B = \{i \in \mathcal{I} \mid a_i = b_i\}, \quad C = \{i \in \mathcal{I} \mid a_i > b_i\},$$

are mutually disjoint and $A \cup B \cup C = \mathcal{I}$ because $\mathbb{R}$ is a totally ordered set. Thus $p(A) + p(B) + p(C) = 1$ by the additivity of the measure p. It follows that exactly one of the following is true: $p(A) = 1$ or $p(B) = 1$ or $p(C) = 1$, since the range of p is $\{0, 1\}$. Thus exactly one of the following is true: $\langle a_i \rangle < \langle b_i \rangle$, $\langle a_i \rangle = \langle b_i \rangle$, and $\langle a_i \rangle > \langle b_i \rangle$.
▲

The rest of the proof of Theorem 5.2 can be done in a similar manner and we leave it to to the reader. ▲

6 Infinitesimals, Finite and Infinitely Large Numbers

6.1 Definition.　*1. We define the sets of **infinitesimal**, **finite**, and **infinitely large** numbers as follows:*

$$\mathcal{I}({}^*\mathbb{C}) = \{x \in {}^*\mathbb{C} : |x| < 1/n \text{ for all } n \in \mathbb{N}\},$$

$$\mathcal{F}({}^*\mathbb{C}) = \{x \in {}^*\mathbb{C} : |x| < n \text{ for some } n \in \mathbb{N}\},$$

$$\mathcal{L}({}^*\mathbb{C}) = \{x \in {}^*\mathbb{C} : |x| > n \text{ for all } n \in \mathbb{N}\},$$

*2. Let $x, y \in {}^*\mathbb{C}$. We say x and y are infinitely close, in symbol $x \approx y$, if $x - y \in \mathcal{I}({}^*\mathbb{C})$. The relation $\approx$ is called **infinitesimal relation** on ${}^*\mathbb{C}$.*

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3. *Let $x \in {}^*\mathbb{C}$ and $r \in \mathbb{C}$. We write $x \rightsquigarrow y$ if $x - r \in \mathcal{I}({}^*\mathbb{C})$. We shall often refer to $\rightsquigarrow$ an* **asymptotic expansion** *of x.*

6.2 Proposition (Basic Properties).

$$\text{(3)} \qquad\qquad {}^*\mathbb{C} = \mathcal{F}({}^*\mathbb{C}) \cup \mathcal{L}({}^*\mathbb{C}),$$

$$\text{(4)} \qquad\qquad \mathcal{F}({}^*\mathbb{C}) \cap \mathcal{L}({}^*\mathbb{C}) = \varnothing,$$

$$\text{(5)} \qquad\qquad \mathcal{I}({}^*\mathbb{C}) \subset \mathcal{F}({}^*\mathbb{C}),$$

$$\text{(6)} \qquad\qquad \mathcal{I}({}^*\mathbb{C}) \cap \mathbb{C} = \{0\},$$

and similarly for ${}^*\mathbb{R}$.

Proof: These results follow directly from the definitions of infinitesimal, finite and infinitely large numbers and the fact that ${}^*\mathbb{R}$ is a totally ordered field. ▲

6.3 Example (Infinitesimals). Let $\nu = \langle a_i \rangle$, where $(a_i) \in \mathbb{C}^{\mathcal{I}}$, $a_i = n$, $n = \max\{m \in \mathbb{N} \mid i \in \mathcal{I}_{m-1} \backslash \mathcal{I}_m\}$. The non-standard number ν is an infinitely large natural number in the sense that $\nu \in {}^*\mathbb{N}$ and $(\forall \varepsilon \in \mathbb{R}_+)(\varepsilon < \nu)$. Indeed, we choose $n \in \mathbb{N}$ such that $\varepsilon \leq n$ and observe that $\mathcal{I}_n \subset \{i \in \mathcal{I} \mid a_i > n \geq \varepsilon\}$. Thus $p(\{i \in \mathcal{I} \mid a_i > \varepsilon\}) = 1$ since $p(\mathcal{I}_n) = 1$. Among other things this example show that ${}^*\mathbb{R}$ and ${}^*\mathbb{C}$ are proper extensions of $\mathbb{R}$ and $\mathbb{C}$, respectively. The numbers ν^n, $\sqrt[n]{\nu}$, $\ln \nu$, e^ν are infinitely large numbers in ${}^*\mathbb{R}$. In contrast, the numbers $1/\nu^n$, $1/\sqrt[n]{\nu}$, $1/\ln \nu$, $e^{-\nu}$ are non-zero infinitesimals in ${}^*\mathbb{R}$. If $r \in \mathbb{R}$, then $r + 1/\nu^n$ is a finite (but not standard) number in ${}^*\mathbb{R}$. Also $e^{i\nu}$ is a finite number in ${}^*\mathbb{C}$ and $e^{i\nu}\nu^2 + i\ln \nu + 5 + 3i$ is an infinitely large number in ${}^*\mathbb{C}$.

Our next goal is to define and study a ring homomorphism st from the ring of finite numbers $\mathcal{F}({}^*\mathbb{C})$ to $\mathbb{C}$, called *standard part mapping*. The standard part mapping is, in a sense, a counterpart of the concept of *limit* in the usual (standard) analysis. In contrast to limit, however, the standard part mapping is applied to non-standard numbers rather than to sequences of standard numbers or functions.

6.4 Definition (Standard Part Mapping). *1. The* **standard part mapping** $\text{st} : \mathcal{F}({}^*\mathbb{R}) \to \mathbb{R}$ *is defined by the formula:*

$$\text{(7)} \qquad\qquad \text{st}(x) = \sup\{r \in \mathbb{R} \mid r < x\}.$$

If $x \in \mathcal{F}(^*\mathbb{R})$, then $\mathrm{st}(x)$ is called the **standard part** *of x.*

The standard part mappings defined in (ii) and (iii) below are extensions of the standard part mapping just defined; we shall keep the same notation, st, *for all.*

2. *The* **standard part mapping** $\mathrm{st} : \mathcal{F}(^*\mathbb{C}) \to \mathbb{C}$ *is defined by the formula* $\mathrm{st}(x + y\,i) = \mathrm{st}(x) + \mathrm{st}(y)\,i$.

3. *The mapping* $\mathrm{st} : {}^*\mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ *is defined by (i) and by* $\mathrm{st}(x) = \pm\infty$ *for* $x \in \mathcal{L}(^*\mathbb{R}_\pm)$, *respectively.*

6.5 Theorem (Standard Part Mapping on Finite Numbers). *1.* **(i)** *Every finite non-standard number $x \in \mathcal{F}(^*\mathbb{C})$ has a unique asymptotic expansion*

$$(8) \qquad\qquad x = \mathrm{st}(x) + dx.$$

where $dx \in \mathcal{I}(^*\mathbb{C})$. Consequently, if $x \in {}^*\mathbb{C}$, then $x \in \mathcal{F}(^*\mathbb{C})$ iff $x = c + dx$ for some $c \in \mathbb{C}$ and some $dx \in \mathcal{I}(^*\mathbb{C})$.

2. *The standard part mapping is a ring homomorphism from $\mathcal{F}(^*\mathbb{C})$ onto $\mathbb{C}$, i.e. for every $x, y \in \mathcal{F}(^*\mathbb{C})$ we have:*

$$(9) \qquad \begin{aligned} &\mathrm{st}(x \pm y) = \mathrm{st}(x) \pm \mathrm{st}(y), \\ &\mathrm{st}(x\,y) = \mathrm{st}(x)\,\mathrm{st}(y), \\ &\mathrm{st}(x/y) = \mathrm{st}(x)/\mathrm{st}(y), \quad \textit{whenever } \mathrm{st}(y) \neq 0. \end{aligned}$$

3. $\mathbb{C}$ *consists exactly of the* **fixed points** *of st in ${}^*\mathbb{C}$, in symbol,*

$$(10) \qquad\qquad \mathbb{C} = \{x \in {}^*\mathbb{C} \mid \mathrm{st}(x) = x\}.$$

Consequently, $\mathrm{st} \circ \mathrm{st} = \mathrm{st}$, *where $\circ$ denotes "composition".*

4. $x \in \mathcal{I}(^*\mathbb{R})$ iff $\mathrm{st}(x) = 0$.

5. *The standard part mapping* st *is an order preserving ring homomorphism from $\mathcal{F}(^*\mathbb{R})$ onto $\mathbb{R}$, where "order preserving" means that if $x, y \in \mathcal{F}(^*\mathbb{R})$, then $x < y$ implies $\mathrm{st}(x) \le \mathrm{st}(y)$ (hence, $x \le y$ implies $\mathrm{st}(x) \le \mathrm{st}(y)$).*

Proof: (i) Suppose, first, that $x \in \mathcal{F}(^*\mathbb{R})$. We have to show that $x - \mathrm{st}(x)$ is infinitesimal. Suppose (on the contrary) that $1/n < |x - \mathrm{st}(x)|$ for some n. In the case $x > \mathrm{st}(x)$, it follows $1/n < x - \mathrm{st}(x)$, contradicting (7). In the case $x < \mathrm{st}(x)$, it follows $1/n < \mathrm{st}(x) - x$, again contradicting (7). To show the uniqueness of (8), suppose that $r + dx = s + dy$ for some $r, s \in \mathbb{R}$ and some $dx, dy \in \mathcal{I}(^*\mathbb{R})$. It follows that $r - s$ is infinitesimal, hence, $r = s$, since the zero is the only infinitesimal in $\mathbb{R}$. The result extends to $\mathcal{F}(^*\mathbb{C})$ directly by the formula in part (ii) of Definition 6.4.

(ii) follows immediately from (i).

(iii) follows immediately from (i) by letting $dx = 0$.

(iv) follows directly from the definition of st.

(v) If $x \approx y$, then it follows $\mathrm{st}(x) = \mathrm{st}(y)$ (regardless whether $x < y$, $x = y$ or $x > y$). Suppose $x < y$ and $x \not\approx y$. It follows $\mathrm{st}(x) + dx < \mathrm{st}(y) + dy$. We have to show that $\mathrm{st}(x) \leq \mathrm{st}(y)$. Suppose (on the contrary) that $\mathrm{st}(x) > \mathrm{st}(y)$. It follows $0 < \mathrm{st}(x) - \mathrm{st}(y) < dy - dx$ implying $\mathrm{st}(x) - \mathrm{st}(y) \approx 0$, hence, $\mathrm{st}(x) = \mathrm{st}(y)$, a contradiction. $\blacktriangle$

6.6 Corollary (An Isomorphism). **(i)** $\mathcal{F}(^*\mathbb{R})/\mathcal{I}(^*\mathbb{R})$ *is ordered field isomorphic to* $\mathbb{R}$ *under the mapping* $q(x) \to \mathrm{st}(x)$, *where* $q : \mathcal{F}(^*\mathbb{R}) \to \mathcal{F}(^*\mathbb{R})/\mathcal{I}(^*\mathbb{R})$ *is the quotient mapping.*

(ii) $\mathcal{F}(^*\mathbb{C})/\mathcal{I}(^*\mathbb{C})$ *is field isomorphic to* $\mathbb{C}$ *under the mapping* $Q(x) \to \mathrm{st}(x)$, *where* $Q : \mathcal{F}(^*\mathbb{C}) \to \mathcal{F}(^*\mathbb{C})/\mathcal{I}(^*\mathbb{C})$ *is the quotient mapping.*

(iii) *The isomorphism described in (ii) is an extension of the isomorphism described in (i).*

We leave the proof to the reader.

6.7 Example. Let $c \in \mathbb{C}$ and let $dx \in \mathcal{I}(^*\mathbb{C})$ be a non-zero infinitesimal. Then we have:

$$\mathrm{st}(c + dx^n) = c,$$
$$\mathrm{st}(dx/|dx|) = \pm 1,$$
$$\mathrm{st}\left(\frac{cdx + 7dx^2 + dx^3}{dx}\right) = \mathrm{st}(c + 7dx + dx^2) = c,$$
$$\mathrm{st}\left(\frac{-3 + 4dx}{dx}\right) = \mathrm{st}(1/dx) \times \mathrm{st}(-3 + 4dx) = (\pm\infty) \times (-3) = \mp\infty,$$

where the choice of the sign $\pm$ depends on whether dx is positive or negative, respectively.

6.8 Definition (Standard Part of a Set). If $\mathcal{A} \subseteq {}^*\mathbb{C}$, we define the **standard part** of $\mathcal{A}$ by

$$(11) \qquad \text{st}[\mathcal{A}] = \{\text{st}(x) \mid x \in \mathcal{A} \cap \mathcal{F}({}^*\mathbb{C})\}.$$

6.9 Lemma. *If $\mathcal{A} \subseteq {}^*\mathbb{C}$, then $\mathcal{A} \cap \mathbb{C} \subseteq \text{st}[\mathcal{A}]$. (A proper inclusion might occur; see the example below.). In particular, we have $\text{st}[{}^*\mathbb{R}] = \mathbb{R}$ and $\text{st}[{}^*\mathbb{C}] = \mathbb{C}$.*

Proof: The inclusion $\mathcal{A} \cap \mathbb{C} \subseteq \text{st}[\mathcal{A}]$ follows directly from part (iii) of Theorem 6.5.

6.10 Example. Consider the set $\mathcal{A} = \{x \in {}^*\mathbb{R} \mid 0 < x < 1\}$. We have $\mathcal{A} \cap \mathbb{C} = \{x \in \mathbb{R} \mid 0 < x < 1\}$. On the other hand, $\text{st}[\mathcal{A}] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$. Indeed, if ϵ is a positive infinitesimal in ${}^*\mathbb{R}$, then ϵ, $1 - \epsilon \in \mathcal{A}$ and $\text{st}(\epsilon) = 0$, and $\text{st}(1 - \epsilon) = 1$.

7 NSA and the Usual Topology on $\mathbb{R}^d$

In what follows we let ${}^*\mathbb{R}^d = {}^*\mathbb{R} \times {}^*\mathbb{R} \times \cdots \times {}^*\mathbb{R}$ (d times). If $x \in {}^*\mathbb{R}^d$, then $x \approx 0$ means that $\|x\|$ is infinitesimal.

7.1 Definition (Monads). If $\mathbb{X} \subseteq \mathbb{R}^d$, then

$$\mu(\mathbb{X}) = \{r + dx \mid r \in \mathbb{X}, \ dx \in {}^*\mathbb{R}^d, \|dx\| \approx 0\}.$$

is called the **monad** of $\mathbb{X}$ in ${}^*\mathbb{R}^d$. If $r \in \mathbb{R}^d$, we shall write simply $\mu(r)$ instead of the more precise $\mu(\{r\})$, i.e.

$$\mu(r) = \{r + dx \mid dx \in \mathcal{I}(\mathbb{R}^d)\}.$$

We observe that $\mu(\mathbb{X}) = \bigcup_{r \in \mathbb{X}} \mu(r)$.

In what follows $\mathcal{T}$ stands for the usual topology on $\mathbb{R}^d$.

7.2 Theorem (Boolean Properties). *The mapping $\mu : \mathcal{T} \to \mathcal{P}({}^*\mathbb{R}^d)$ is a Boolean homomorphism. Also μ preserves the arbitrary unions in the sense that $\mu\left(\bigcup_{\lambda \in \Lambda} \Omega_\lambda\right) = \bigcup_{\lambda \in \Lambda} \mu(\Omega_\lambda)$ for any set Λ and any family of open sets $\{\Omega_\lambda\}_{\lambda \in \Lambda}$.*

7.3 Theorem (The Usual Topology on $\mathbb{R}^d$). *Let $\mathbb{X} \subseteq \mathbb{R}^d$. Then:*

1. *A set $\mathbb{X}$ is **open** in $\mathbb{R}^d$ iff $\mu(\mathbb{X}) \subseteq {}^*\mathbb{X}$.*

2. *$\mathbb{X}$ is **compact** in $\mathbb{R}^d$ iff ${}^*\mathbb{X} \subseteq \mu(\mathbb{X})$.*

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8 Non-Standard Smooth Functions

8.1 Definition (Non-Standard Smooth Functions). *Let Ω is an open set of $\mathbb{R}^d$. Then:*

1. *The ring (algebra) of the* **non-standard smooth functions** *is defined the factor ring*
$$^*\mathcal{E}(\Omega) = \mathcal{E}(\Omega)^{\mathcal{I}}/\sim,$$
where $(f_i) \sim (g_i)$ if $f_i = g_i$ in $\mathcal{E}(\Omega)$ **for almost all** *i in the sense that*
$$p\left(\{i \mid f_i = g_i\}\right) = 1.$$

   We denote by $\langle f_i \rangle \in {}^*\mathcal{E}(\Omega)$ the equivalence class determined by (f_i).

2. *The algebraic operations and partial differentiation in ${}^*\mathcal{E}(\Omega)$ is inherited from ${}^*\mathcal{E}(\Omega)$. For example, $\partial^\alpha \langle f_i \rangle = \langle \partial^\alpha f_i \rangle$.*

3. *The mapping $f \to {}^*f$ defines an embedding $\mathcal{E}(\Omega) \hookrightarrow {}^*\mathcal{E}(\Omega)$ by the constant families, i.e. $f_i = f$ for all $i \in \mathcal{I}$. We say that ${}^*f$ is the* **non-standard extension** *of f.*

4. **(iv)** *Every $\langle f_i \rangle \in {}^*\mathcal{E}(\Omega)$ is a* **pointwise mapping** *of the form $\langle f_i \rangle : {}^*\Omega \to {}^*\mathbb{C}$, where $\langle f_i \rangle(\langle x_i \rangle) = \langle f_i(x_i) \rangle$ and*
$$^*\Omega = \{\langle x_i \rangle \in {}^*\mathbb{R}^d \mid x_i \in \Omega \quad a.e. \ \},$$

   is the **non-standard extension** *of Ω.*

5. **(v)** *Let $X \subseteq \mathcal{E}$. The non-standard extension ${}^*X$ of X is defined by*
$$^*X = \{\langle f_i \rangle \in {}^*\mathcal{E}(\Omega) \mid f_i \in X \ a.e. \ \}.$$

   In particular,
$$^*\mathcal{D}(\Omega) = \{\langle f_i \rangle \in {}^*\mathcal{E}(\Omega) \mid f_i \in \mathcal{D}(\Omega) \ a.e. \ \}.$$

8.2 Proposition. *${}^*\mathcal{E}(\Omega)$ is a differential algebra over the field ${}^*\mathbb{C}$.*

8.3 Definition (Sup and Support). *Let $\langle f_i \rangle \in {}^*\mathcal{E}(\Omega)$ and let $K \subset\subset \Omega$. Then*

1. $\sup_{x \in {}^*K} |\langle f_i \rangle(x)| = \langle \sup_{x \in K} |f_i(x)| \rangle.$

2. $\operatorname{supp}\langle f_i \rangle = \langle \operatorname{supp}(f_i) \rangle.$

We shall refer to these as **internal sup** *and* **internal support** *of* $\langle f_i \rangle$, *respectively.*

8.4 Proposition. *Let* $f \in {}^*\mathcal{E}(\Omega)$. *Then:*

1. $(\forall K \subset\subset \Omega)(\sup_{x \in {}^*K} f(x) \in {}^*\mathbb{R}).$

2. $\operatorname{supp}(f)$ *is a closed set of* ${}^*\mathbb{R}$ *in the interval topology of* ${}^*\mathbb{R}$.

8.5 Lemma (Characterizations). *Let* $f \in {}^*\mathcal{E}(\Omega)$ *and* $\operatorname{supp}(f)$ *denote the (internal) support of* f *in* ${}^*\Omega$. *Then* **the following are equivalent:**

(i) $\operatorname{supp}(f) \subset \mu(\Omega)$.

(ii) $\exists K \subset\subset \Omega$ *such that* $\operatorname{supp}(f) \subseteq {}^*K$.

(iii) *There exists an open relatively compact subset* $\mathcal{O}$ *of* Ω *such that* $f \in {}^*\mathcal{D}(\mathcal{O})$ *(The latter implies* $f(x) = 0$ *for all* $x \in {}^*(\Omega \setminus \mathcal{O})$.)

8.6 Definition (Compact Support). *Let* $\mathcal{X} \subseteq {}^*\mathcal{E}(\Omega)$. *We denote*

$$\mathcal{X}_c = \{f \in \mathcal{X} \mid \operatorname{supp}(f) \subset \mu(\Omega)\}.$$

In particular, we have:

$$(12) \qquad {}^*\mathcal{D}_c(\Omega) = \{f \in {}^*\mathcal{D}(\Omega) \mid \operatorname{supp}(f) \subset \mu(\Omega)\},$$

$$(13) \qquad \mathcal{X}_c = {}^*\mathcal{D}_c(\Omega) \cap \mathcal{X},$$

$$(14) \qquad {}^*\mathcal{D}_c(\Omega) = {}^*\mathcal{E}_c(\Omega) = \{f \in {}^*\mathcal{E}(\Omega) \mid \operatorname{supp}(f) \subset \mu(\Omega)\}.$$

8.7 Lemma (Characterizations). *Let* $f \in {}^*\mathcal{E}(\Omega)$. *Then the following are equivalent:*

1. $(\forall x \in \mu(\Omega))\,[f(x) \in \mathcal{M}_\rho({}^*\mathbb{C})].$

2. $(\forall K \subset\subset \Omega)(\exists n \in \mathbb{N})(\sup_{x \in {}^*K} |f(x)| \leq \rho^{-n}).$

3. **(iii)** $(\forall K \subset\subset \Omega)(\forall n \in {}^*\mathbb{N} \setminus \mathbb{N})(\sup_{x \in {}^*K} |f(x)| \leq \rho^{-n}).$

8.8 Lemma (Characterizations). *Let* $f \in {}^*\mathcal{E}(\Omega)$. *Then the following are equivalent:*

1. $(\forall x \in \mu(\Omega))\,[f(x) \in \mathcal{N}_\rho({}^*\mathbb{C})].$

2. $(\forall K \subset\subset \Omega)(\forall n \in \mathbb{N})(\sup_{x \in {}^*K} |f(x)| \leq \rho^{n}).$

3. $(\forall K \subset\subset \Omega)(\exists n \in {}^*\mathbb{N} \setminus \mathbb{N})(\sup_{x \in {}^*K} |f(x)| \leq \rho^{n}).$

9   Local Properties of $^*\mathcal{E}(\Omega)$

In what follows $\mathcal{T}_d$ stands for the usual topology on $\mathbb{R}^d$ and we denote by $(\mathbb{R}^d, \mathcal{T}_d)$ the corresponding topological space. Also we denote by $^*\mathcal{T}_d$ the order topology of $^*\mathbb{R}^d$ (more precisely, $^*\mathcal{T}_d$ stands for the product topology on $^*\mathbb{R}^d$ generated by the order topology on $^*\mathbb{R}$). We denote by $(^*\mathbb{R}^d, {}^*\mathcal{T}_d)$ the corresponding topological space.

The purpose of this section is to show that the collection of the non-standard spaces $\{^*\mathcal{E}(\Omega)\}_{\Omega \in {}^*\mathcal{T}_d}$ (Section 8) is a *sheaf* on $(^*\mathbb{R}^d, {}^*\mathcal{T}_d)$, but in contrast, $\{^*\mathcal{E}(\Omega)\}_{\Omega \in {}^*\mathcal{T}_d}$ is only a *presheaf* on $(\mathbb{R}^d, \mathcal{T}_d)$. For the relevent terminology we refer to A. Kaneko [39].

9.1 Theorem (Non-Standard Sheaf). *The collection $\{^*\mathcal{E}(\Omega)\}_{\Omega \in {}^*\mathcal{T}_d}$ is a **sheaf of differential rings on** $(^*\mathbb{R}^d, {}^*\mathcal{T}_d)$ under the usual pointwise restriction in $^*\mathcal{E}(\Omega)$.*

Proof: From the (standard) functional analysis we know that the collection $\{\mathcal{E}(\Omega)\}_{\Omega \in \mathcal{T}_d}$ is a sheaf of differential rings on $\mathbb{R}^d$ in the sense that $f \in \mathcal{E}(\Omega)$ and $\mathcal{O} \subseteq \Omega$ implies $f|\mathcal{O} \in \mathcal{E}(\mathcal{O})$ for every $\Omega, \mathcal{O} \in \mathcal{T}_d$. Thus $f \in {}^*\mathcal{E}(\Omega)$ implies $f|\mathcal{O} \in {}^*\mathcal{E}(\mathcal{O})$ for every $\Omega \in \mathcal{T}_d$ and $\mathcal{O} \in {}^*\mathcal{T}_d$ such that $\mathcal{O} \subseteq {}^*\Omega$ by Transfer Principle (Theorem 4.8). ▲

9.2 Corollary (Non-Standard Support). *Let $f \in {}^*\mathcal{E}(\Omega)$ and $\mathrm{supp}(f)$ be the support of f (Definition 8.3). Then $\mathrm{supp}(f)$ is a closed set of $^*\Omega$ in the topology $^*\mathcal{T}_d$ on $^*\mathbb{R}^d$.*

Proof: The result follows (also) by Transfer Principle (or directly from the above theorem).
▲

Let $\mathcal{O}, \Omega \in \mathcal{T}_d$ be two (standard) open sets such that $\mathcal{O} \subseteq \Omega$ and $f \in {}^*\mathcal{E}(\Omega)$. We define the restriction $f \upharpoonright \mathcal{O} = f \,|{}^*\mathcal{O}$.

9.3 Theorem (Standard Presheaf). *The collection $\{^*\mathcal{E}(\Omega)\}_{\Omega \in \mathcal{T}_d}$ is a **presheaf of differential rings on** $(\mathbb{R}^d, \mathcal{T}_d)$ under the restriction $\upharpoonright$ in the sense that:*

1. $(\forall \Omega \in \mathcal{T}_d)(\forall f \in {}^*\mathcal{E}(\Omega))(f \upharpoonright \Omega = f)$.

2. $(\forall \Omega_1, \Omega_2, \Omega \in \mathcal{T}_d)(\forall f \in {}^*\mathcal{E}(\Omega))(\Omega_1 \subseteq \Omega_2 \subseteq \Omega \text{ implies } (f \upharpoonright \Omega_2) \upharpoonright \Omega_1 = f \upharpoonright \Omega_1$.

3. $(\forall \Omega, \mathcal{O} \in \mathcal{T}_d)(\forall f, g \in {}^*\mathcal{E}(\Omega))(\mathcal{O} \subseteq \Omega \Rightarrow (f + g) \upharpoonright \mathcal{O} = f \upharpoonright \mathcal{O} + g \upharpoonright \mathcal{O})$.

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4. $(\forall \Omega, \mathcal{O} \in \mathcal{T}_d)(\forall f, g \in {}^*\mathcal{E}(\Omega)) \, (\mathcal{O} \subseteq \Omega \Rightarrow (fg) \upharpoonright \mathcal{O} = (f \upharpoonright \mathcal{O})(g \upharpoonright \mathcal{O})).$

5. $(\forall \Omega, \mathcal{O} \in \mathcal{T}_d)(\forall f \in {}^*\mathcal{E}(\Omega))(\forall \alpha \in \mathbb{N}_0^d) \, (\mathcal{O} \subseteq \Omega \Rightarrow (\partial^\alpha f) \upharpoonright \mathcal{O} = (\partial^\alpha (f \upharpoonright \mathcal{O})).$

Proof. (1) $f \upharpoonright \Omega = f \,|\, {}^*\Omega = f$ (as required) since ${}^*\Omega$ is the domain of f.

(2) $(f \upharpoonright \Omega_2) \upharpoonright \Omega_1 = (f \,|\, {}^*\Omega_2) \,|\, {}^*\Omega_1 = f \,|\, {}^*\Omega_1 = f \upharpoonright \Omega_1$ (as required) since ${}^*\Omega_1 \subseteq {}^*\Omega_2 \subseteq {}^*\Omega$. The rest of the properties are proved similarly and we leave them to the reader. ∎

9.4 Remark (A Counter Example). The next example shows that the collection $\{{}^*\mathcal{E}(\Omega)\}_{\Omega \in \mathcal{T}_d}$ **is not a sheaf** on $(\mathbb{R}^d, \mathcal{T}_d)$ under the restriction $f \upharpoonright \mathcal{O} = f \,|\, {}^*\mathcal{O}$. Indeed, let $\Omega = \mathbb{R}_+$ and $\Omega_n = (0, n)$ for $n \in \mathbb{N}$. Let $\varphi \in \mathcal{D}(\mathbb{R}_+), \varphi \neq 0$, and let ν be an infinitely large number in ${}^*\mathbb{R}_+$ (see Example 6.3). We define $f(x) = {}^*\varphi(x - \nu)$ for all $x \in {}^*\mathbb{R}_+$. It is clear that $\bigcup_{n \in \mathbb{N}}(0, n) = \mathbb{R}_+$ and $f \upharpoonright (0, n) = f \,|\, {}^*(0, n) = 0$ for all n. Yet, $f \upharpoonright \mathbb{R}_+ = f \,|\, {}^*\mathbb{R}_+ = f \neq 0$.

Our conclusion is that in order to convert the non-standard smooth functions ${}^*\mathcal{E}(\Omega)$ into an algebra of generalized functions, **we have to perform a factorization of the space** ${}^*\mathcal{E}(\Omega)$. A general method for such factorization will be presented in Section 23.

10 Asymptotic Fields

Let κ be an infinite cardinal and let ${}^*\mathbb{C}$ be a κ-saturated non-standard extension of the field of the complex numbers $\mathbb{C}$. We describe those *algebraically closed subfields* $\widehat{\mathcal{M}}$ of ${}^*\mathbb{C}$ which are *Cantor κ-complete*. The fields $\widehat{\mathcal{M}}$ are constructed as factor rings of a given convex subring $\mathcal{M}$ of ${}^*\mathbb{C}$. We call these fields $\mathcal{M}$**-asymptotic fields** and their elements $\mathcal{M}$**-asymptotic numbers**. Our *asymptotic field construction* can be viewed as a generalization of A. Robinson's theory of asymptotic numbers (Lightstone & Robinson [57]). In our approach Robinson field ${}^\rho\mathbb{C}$ appears as a subfield of ${}^*\mathbb{C}$. We also generalize some more recent results in (T. Todorov and R. Wolf [97]) on the A. Robinson field ${}^\rho\mathbb{R}$. A construction similar to the presented here appears in the H. Vernaeve Ph.D. Thesis [100] (for a comparison see the equivalence relation $\sim$ defined on p. 87, Sec. 3.6, altered by the additional condition used in Lemma 3.32 on p. 89).

Algebraically closed non-archimedean fields had been studied in model theory of fields (Ribenboim [**?**]) in the form of generalized power series

(Hahn [33]). These fields are usually defined without connection with non-standard analysis. In contrast, in our approach many of these fields are defined in the framework of $^*\mathbb{C}$ and they can be embedded as subfields of $^*\mathbb{C}$. In particular, we show that the Hahn field of generalized power series $\mathbb{C}(t^{\mathbb{R}})$ and the logarithmic-exponential field $\mathbb{R}((t))^{LE}$ (Marker, Messmer & Pillay [62]) are subfields of $^*\mathbb{C}$. For that reason we hope that our *asymptotic field construction* might facilitate the communication between the mathematicians working in non-standard analysis and those working in model theory of fields.

The **main purpose** of our algebraic approach however is to support the **theory of $\mathcal{M}$-asymptotic functions** $\widehat{\mathcal{M}(\Omega)}$ on an open set $\Omega \subseteq \mathbb{R}^d$ presented in Section 20. Each $\widehat{\mathcal{M}(\Omega)}$ is an algebra of generalized functions over field of scalars $\widehat{\mathcal{M}}$. We show that each $\widehat{\mathcal{M}(\Omega)}$ contains a copy of the space of Schwartz distributions $\mathcal{D}'(\Omega)$ and in this sense $\widehat{\mathcal{M}(\Omega)}$ are algebras of Colombeau type (Colombeau [10]-[12]). In particular, we show that the space of non-standard functions $^*\mathcal{E}(\Omega)$ also contains a copy of $\mathcal{D}'(\Omega)$. Here $\mathcal{E}(\Omega) = \mathcal{C}^\infty(\Omega)$ stands for the usual class of $\mathcal{C}^\infty$-functions on Ω and $^*\mathcal{E}(\Omega)$ is its non-standard extension.

11 Convex Rings in $^*\mathbb{C}$

In what follows $^*\mathbb{R}$ stands for a non-standard extension of the field of real numbers $\mathbb{R}$ and $^*\mathbb{C} = {}^*\mathbb{R}(i)$. If $S \subseteq {}^*\mathbb{C}$, then $\mathcal{I}(S), \mathcal{F}(S)$ and $\mathcal{L}(S)$ stand for the sets of infinitesimal, finite and infinitely large numbers in S, respectively.

11.1 Definition (Convex Rings)**.** We say that a subring $\mathcal{M}$ of $^*\mathbb{C}$ is **convex** in $^*\mathbb{C}$ if $(\forall z \in {}^*\mathbb{C})(\forall \zeta \in \mathcal{M})(|z| \leq |\zeta| \Rightarrow z \in \mathcal{M})$. We denote by $\mathcal{M}_0$ the set of all **non-invertible elements** of $\mathcal{M}$, i.e.

(15) $$\mathcal{M}_0 = \{z \in \mathcal{M} \mid z = 0 \text{ or } 1/z \notin \mathcal{M}\}.$$

11.2 Lemma (Convex Rings)**.** *Let $\mathcal{M}$ be a convex subring of $^*\mathbb{C}$. Then:*

(i) *$\mathcal{M}$ contains a copy of the ring $\mathcal{F}(^*\mathbb{C})$ of the finite elements of $^*\mathbb{C}$. Consequently, $\mathcal{M}$ contains a copy $\mathbb{C}$. We summarize all these as $\mathbb{C} \subset \mathcal{F}(^*\mathbb{C}) \subseteq \mathcal{M} \subseteq {}^*\mathbb{C}$.*

(ii) ${}^*\mathbb{R} \cap \mathcal{M}$ *is a real ring and* ${}^*\mathbb{R} \cap \mathcal{M}$ *is convex in* ${}^*\mathbb{R}$ *in the sense that if* $x \in {}^*\mathbb{R}$ *and* $y \in {}^*\mathbb{R} \cap \mathcal{M}$*, then* $|x| \leq |y|$ *implies* $x \in {}^*\mathbb{R} \cap \mathcal{M}$*. Also,* $\mathcal{F}({}^*\mathbb{R}) \subseteq {}^*\mathbb{R} \cap \mathcal{M}$*.*

(iii) $({}^*\mathbb{R} \cap \mathcal{M})(i) = \mathcal{M}$*, where* $({}^*\mathbb{R} \cap \mathcal{M})(i) =: \{x + iy : x, y \in {}^*\mathbb{R} \cap \mathcal{M}\}$*.*

(iv) $\mathcal{M}$ *is an archimedean ring* iff $\mathcal{M} = \mathcal{F}({}^*\mathbb{C})$ iff ${}^*\mathbb{R} \cap \mathcal{M} = \mathcal{F}({}^*\mathbb{R})$*.*

(v) $\mathcal{M}$ *is a field* iff $\mathcal{M} = {}^*\mathbb{C}$ iff ${}^*\mathbb{R} \cap \mathcal{M}$ *is a field* iff ${}^*\mathbb{R} \cap \mathcal{M} = {}^*\mathbb{R}$*.*

Proof. (i) Observe that ${}^*\mathbb{R} \cap \mathcal{M}$ is a totally ordered ring (as a subring of ${}^*\mathbb{R}$) and thus it contains the ring of the integers $\mathbb{Z}$. Thus $\mathcal{M}$ contains $\mathbb{Z}$. With this in mind, suppose $z \in \mathcal{F}({}^*\mathbb{C})$, i.e. $|z| \leq n$ for some $n \in \mathbb{N}$. The latter implies $z \in \mathcal{M}$ (as desired) by the convexity of $\mathcal{M}$ since $n \in \mathcal{M}$.

(ii) follows immediately from (i).

(iii) $({}^*\mathbb{R} \cap \mathcal{M})(i) \subseteq \mathcal{M}$ holds because both ${}^*\mathbb{R}$ and $\mathcal{M}$ are rings and $\mathbb{C} \subset \mathcal{M}$ by (i). To show that $({}^*\mathbb{R} \cap \mathcal{M})(i) \supseteq \mathcal{M}$ observe that $\mathcal{M} \subseteq \Re e(\mathcal{M}) + i\Im m(\mathcal{M})$, where $\Re e(\mathcal{M}) = \{\Re e(z) : z \in \mathcal{M}\}$ and $\Im m(\mathcal{M}) = \{\Im m(z) : z \in \mathcal{M}\}$. It remains to show that $\Re e(\mathcal{M}) = \Im m(\mathcal{M}) = {}^*\mathbb{R} \cap \mathcal{M}$. Indeed, ${}^*\mathbb{R} \cap \mathcal{M}$ is (trivially) a subset of $\Re e(\mathcal{M})$. Also, ${}^*\mathbb{R} \cap \mathcal{M} \subseteq \Im m(\mathcal{M})$, because $y \in {}^*\mathbb{R} \cap \mathcal{M}$ implies $iy \in \mathcal{M}$ which implies $\Im m(iy) = y$. Finally, $\Re e(\mathcal{M}) \subseteq \mathcal{M}$ and $\Im m(\mathcal{M}) \subseteq \mathcal{M}$ by the convexity of $\mathcal{M}$.

(iv) Notice that $\mathcal{F}({}^*\mathbb{C})$ is an archimedean ring (by the definition of $\mathcal{F}({}^*\mathbb{C})$). Suppose (on the contrary) that there exists $\lambda \in \mathcal{M} \setminus \mathcal{F}({}^*\mathbb{C})$. That means that λ is infinitely large number, i.e. $n < |\lambda|$ for all $n \in \mathbb{N}$. Thus $\mathcal{M}$ is a non-archimedean ring.

(v) Let $\mathcal{M}$ be a field and suppose (on the contrary) that there exists $\lambda \in {}^*\mathbb{C} \setminus \mathcal{M}$. We choose $\zeta \in \mathcal{M}, \zeta \neq 0$, and observe that $|\lambda| > |\zeta|$ by the convexity of $\mathcal{M}$. The latter implies $|1/\lambda| < |1/\zeta|$ which implies $1/\lambda \in \mathcal{M}$ again by the convexity of $\mathcal{M}$ since $1/\zeta \in \mathcal{M}$. Thus $\lambda \in \mathcal{M}$ since $\mathcal{M}$ is a field, a contradiction. This reverse is clear since ${}^*\mathbb{C}$ is a field. $\square$

11.3 Lemma (Convex Ideals)**.** *Let* $\mathcal{M}$ *be a convex subring of* ${}^*\mathbb{C}$ *and let* $\mathcal{M}_0$ *be the set of the non-invertible elements of* $\mathcal{M}$ *(Definition 14.1). Then*

(i) *If* $z \in {}^*\mathbb{C}$*,* $z \neq 0$*, then* $z \in \mathcal{M}_0$ iff $1/z \notin \mathcal{M}$*. Consequently, we have* $\mathcal{M}_0 = \{z \in {}^*\mathbb{C} \mid z = 0 \text{ or } 1/z \notin \mathcal{M}\}$*.*

(ii) $\mathcal{M}_0$ *consists of* **infinitesimals only***, i.e.* $\mathcal{M}_0 \subseteq \mathcal{I}({}^*\mathbb{C})$*.*

(iii) $\mathcal{M}_0$ is a **convex maximal ideal** in $\mathcal{M}$, i.e. $\mathcal{M}_0$ is a maximal ideal in $\mathcal{M}$ such that if $z \in \mathcal{M}$ and $h \in \mathcal{M}_0$, then $|z| \leq |h|$ implies $z \in \mathcal{M}_0$. Consequently, $\mathcal{M}$ is a **local ring** ($\mathcal{M}_0$ is the only maximal ideal in $\mathcal{M}$.)

(iv) $^*\mathbb{R} \cap \mathcal{M}_0$ is a convex maximal ideal in $^*\mathbb{R} \cap \mathcal{M}$. Consequently, $^*\mathbb{R} \cap \mathcal{M}$ is also a **local real ring** (i.e. $^*\mathbb{R} \cap \mathcal{M}_0$ is the only maximal ideal in $^*\mathbb{R} \cap \mathcal{M}$.)

(v) The sets $\mathcal{M}_0$, $\mathcal{M} \setminus \mathcal{M}_0$ and $^*\mathbb{C} \setminus \mathcal{M}$ are **disconnected** in the sense that

$$(\forall z_1 \in \mathcal{M}_0)(\forall z_2 \in \mathcal{M} \setminus \mathcal{M}_0)(\forall z_3 \in {}^*\mathbb{C} \setminus \mathcal{M})(|z_1| < |z_2| < |z_3|).$$

Proof. (i) Let $z \in {}^*\mathbb{C}, z \neq 0$. We have $1/z \notin \mathcal{M} \Rightarrow |1/z| > 1 \Rightarrow |z| < 1 \Rightarrow z \in \mathcal{M}$ by the convexity of $\mathcal{M}$. The latter (along with $1/z \notin \mathcal{M}$) implies $z \in \mathcal{M}_0$ as required.

(ii) We observe that $x \in \mathcal{M}_0 \setminus \mathcal{I}(^*\mathbb{C})$ implies $1/x \notin \mathcal{M}$, which implies $1/x \notin \mathcal{F}(^*\mathbb{C})$ since $\mathcal{F}(^*\mathbb{C}) \subseteq \mathcal{M}$ by (i) of Lemma 11.2. The latter implies $1/x \in \mathcal{L}(^*\mathbb{C})$, which implies $x \in \mathcal{I}(^*\mathbb{C})$, a contradiction.

(iii) Let $z \in \mathcal{M}$ and $h \in \mathcal{M}_0$ and suppose (on the contrary) that $zh \notin \mathcal{M}_0$. It follows that $1/zh \in \mathcal{M}$ which implies $1/h \in \mathcal{M}$ thus $h \notin \mathcal{M}_0$ by (i), a contradiction. The fact that $\mathcal{M}_0$ is closed under the addition follows from (i). Indeed, let $h_1, h_2 \in \mathcal{M}_0$. If $h_1 = 0$ or $h_2 = 0$ there is nothing to prove. Let $h_1 \neq 0$ or $h_2 \neq 0$ and suppose (on the contrary) that $h_1 + h_2 \notin \mathcal{M}_0$. It follows $1/(h_1 + h_2) \in \mathcal{M}$ (by the definition of $\mathcal{M}_0$ (15)), which implies $h_1/(h_1 + h_2) \in \mathcal{M}_0$ (by what was proved above) implying $(h_1 + h_2)/h_1 \notin \mathcal{M}$ (by the definition of $\mathcal{M}_0$ again) implying $h_2/h_1 \notin \mathcal{M}$. The latter implies $h_1/h_2 \in \mathcal{M}_0$ by (i). Similarly, we conclude that $h_2/h_1 \in \mathcal{M}_0$. Thus $1 \in \mathcal{M}_0$, a contradiction. The ideal $\mathcal{M}_0$ is maximal (and $\mathcal{M}$ is a local ring) because $\mathcal{M}_0$ consists of all non-invertible elements of $\mathcal{M}$. To show the convexity of $\mathcal{M}_0$, observe that $h = 0$ implies (trivially) $z = 0$. Let $h \neq 0$ and suppose (on the contrary) that $z \notin \mathcal{M}_0$, i.e. $1/z \in \mathcal{M}$. The latter implies $1/h \in \mathcal{M}$ by the convexity of $\mathcal{M}$ which implies $h \notin \mathcal{M}_0$, a contradiction.

(iv) follows directly from (iii).

(v) follows directly from the convexity of both $\mathcal{M}$ and $\mathcal{M}_0$ and (i). $\quad\square$

12 Examples of Convex Rings

We present several examples for convex subrings of $^*\mathbb{C}$ and their maximal ideals (Section 11).

12.1 Definition (Generating Sequences). **(i)** A **decreasing sequence** (δ_n) of infinitely large positive numbers in $^*\mathbb{R}$ is called **generating** if

(a) For every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $2\delta_m \leq \delta_n$.

(b) For every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $\delta_m^2 \leq \delta_n$.

(ii) An **increasing sequence** (λ_n) of infinitely large positive numbers in $^*\mathbb{R}$ is called **generating** if

(a) For every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $2\lambda_n \leq \lambda_m$.

(b) For every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $\lambda_n^2 \leq \lambda_m$.

The next lemma is useful for generating examples of convex subrings (see below).

12.2 Lemma (Generated Rings). **(i)** *Let (δ_n) be a decreasing generating sequence. Then*

$$\mathcal{M} = \{z \in {}^*\mathbb{C} : |z| < \delta_n \text{ for all } n \in \mathbb{N}\},$$

is a convex subring of $^*\mathbb{C}$ and its (unique) maximal ideal is given by

$$\mathcal{M}_0 = \{z \in {}^*\mathbb{C} : |z| \leq 1/\delta_n \text{ for some } n \in \mathbb{N}\}.$$

We say that $\mathcal{M}$ is generated by the sequence (δ_n).

(ii) *Let (λ_n) be an increasing generating sequence. Then*

$$\mathcal{M} = \{z \in {}^*\mathbb{C} : |z| \leq \lambda_n \text{ for some } n \in \mathbb{N}\},$$

is a convex subring of $^*\mathbb{C}$ and its (unique) maximal ideal is given by

$$\mathcal{M}_0 = \{z \in {}^*\mathbb{C} : |z| < 1/\lambda_n \text{ for all } n \in \mathbb{N}\}.$$

We say that $\mathcal{M}$ is generated by the sequence (λ_n).

Proof. The proof is immediate and we leave the proof to the reader. $\square$

Here are **several example** of convex subrings of ${}^*\mathbb{C}$ and their maximal ideals. All but the first example are about non-archimedean rings.

12.3 Example (Finite Numbers)**.** The ring of the finite complex non-standard numbers $\mathcal{F}({}^*\mathbb{C})$ is a convex subring of ${}^*\mathbb{C}$. Its maximal ideal is the set of infinitesimals $\mathcal{I}({}^*\mathbb{C})$. We shall often write $\mathcal{F}$ and $\mathcal{I}$ instead of $\mathcal{F}({}^*\mathbb{C})$ and $\mathcal{I}({}^*\mathbb{C})$, respectively.

12.4 Example (Multiple-Logarithmic Rings)**.** Let ρ be a positive infinitesimal in ${}^*\mathbb{R}$ and let

$$\mathcal{L}_\rho = \{z \in {}^*\mathbb{C} : \ |z| < \log_n (1/\rho) \ \text{ for all } \ n \in \mathbb{N}\}.$$

Here $\log_1(x) = {}^* \ln x$, where ${}^* \ln x$ is the non-standard extension of the usual natural logarithmic function $\ln x$, and we also define $\log_2 = {}^* \ln \circ {}^* \ln$, and $\log_n = {}^* \ln \circ {}^* \ln \circ \cdots \circ {}^* \ln$ (n times). Notice that $(\log_n (1/\rho))$ is a decreasing generating sequence in ${}^*\mathbb{R}$ (Definition 12.1). Indeed, for every n we have $\lim_{x\to\infty} \frac{\log_{n+1} x}{\log_n x} = 0$ by the L'Hopital rule. Thus $\frac{\log_{n+1} (1/\rho)}{\log_n (1/\rho)} \approx 0$ implying $2 \log_{n+1} (1/\rho) < \log_n (1/\rho)$ for all n. Similarly, $\lim_{x\to\infty} \frac{(\log_{n+1} x)^2}{\log_n x} = 0$ by L'Hopital rule. Thus $(\log_{n+1} (1/\rho))^2 < \log_n (1/\rho)$ for all n. Consequently, $\mathcal{L}_\rho({}^*\mathbb{C})$ is a convex subring of ${}^*\mathbb{C}$ by Lemma 12.2. For its maximal ideal we have
$$\mathcal{L}_{\rho,0} = \{z \in {}^*\mathbb{C} : \ |z| \leq \frac{1}{\log_n(1/\rho)} \ \text{ for some } \ n \in \mathbb{N}\}.$$

Let ν be an infinitely large number in ${}^*\mathbb{N}$. Then $\log_\nu (1/\rho)$ is a typical element of $\mathcal{L}_\rho \setminus \mathcal{L}_{\rho,0}$. Here $\log_\nu (1/\rho) = {}^* \ln ({}^*\ln (\ldots {}^* \ln ({}^* \ln (1/\rho) \ldots)))$ (ν times) hence, the name **multiple logarithmic ring** for $\mathcal{L}_\rho$.

12.5 Example (Logarithmic Rings)**.** Let ρ be (as before) a positive infinitesimal in ${}^*\mathbb{R}$. We define

(16) $$\mathcal{F}_\rho = \{z \in {}^*\mathbb{C} : \ |z| < 1/\sqrt[n]{\rho} \ \text{ for all } \ n \in \mathbb{N}\}.$$

We observe that $(1/\sqrt[n]{\rho})$ is a decreasing generating sequence in ${}^*\mathbb{R}$ (Definition 12.1) because $2/\sqrt[n+1]{\rho} < 1/\sqrt[n]{\rho}$ and also (trivially) $(1/\sqrt[2n]{\rho})^2 \leq 1/\sqrt[n]{\rho}$ for all n. Thus $\mathcal{F}_\rho$ is a covex subring of ${}^*\mathbb{C}$ by Lemma 12.2. For its maximal ideal we have

(17) $$\mathcal{I}_\rho = \{z \in {}^*\mathbb{C} : \ |z| \leq \sqrt[n]{\rho} \ \text{ for some } \ n \in \mathbb{N}\}.$$

We call $\mathcal{F}_\rho$ **logarithmic rings** because $\ln \rho$ is a typical element of $\mathcal{F}_\rho$. The numbers in $\mathcal{F}_\rho$ are called **logarithmic numbers** or ρ**-finite numbers**. The numbers in $\mathcal{I}_\rho$ are called ρ**-infinitesimal numbers**. Notice as well that $\sqrt[\nu]{\rho} \in \mathcal{F}_\rho$ for every $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$.

12.6 Example (Robinson Rings)**.** Let ρ be (as before) a positive infinitesimal in ${}^*\mathbb{R}$. The ring of the the ρ**-moderate non-standard numbers** is defined by

$$(18) \qquad \mathcal{M}_\rho = \{z \in {}^*\mathbb{C} : \ |z| \le \rho^{-n} \ \text{ for some } n \in \mathbb{N}\},$$

(Robinson [76]). Notice that (ρ^{-n}) is an increasing generating sequence in ${}^*\mathbb{R}$ (Definition 12.1). Thus $\mathcal{M}_\rho$ is a covex subring of ${}^*\mathbb{C}$ by Lemma 12.2. For its maximal ideal we have

$$(19) \qquad \mathcal{N}_\rho = \{z \in {}^*\mathbb{C} : \ |z| \le \rho^n \ \text{ for all } n \in \mathbb{N}\}.$$

We call the numbers in $\mathcal{N}_\rho$ ρ**-negligible** (or **iota numbers**). The numbers in ${}^*\mathbb{C} \setminus \mathcal{M}_\rho$ are called **mega numbers**. If $x \in \mathbb{R}$, then ρ^x is a typical element of $\mathcal{M}_\rho \setminus \mathcal{N}_\rho$.

12.7 Example (Logarithmic-Exponential Rings)**.** Let ρ be (as before) a positive infinitesimal in ${}^*\mathbb{R}$ and let

$$\mathcal{E}_\rho = \{z \in {}^*\mathbb{C} : \ |z| \le \exp_n(1/\rho) \ \text{ for some } n \in \mathbb{N}\}.$$

Here $\exp_1(x) = {}^*e^x$ is the non-standard extension of the usual natural exponential function e^x, and we also let $\exp_2 = \exp_1 \circ \exp_1$, and $\exp_n = \exp_1 \circ \exp_1 \circ \cdots \circ \exp_1$ (n times). We observe that $(\exp_n(1/\rho))$ is an increasing generating sequence in ${}^*\mathbb{R}$ (Definition 12.1). Indeed, for every n we have $\lim_{x \to \infty} \frac{\exp_n x}{\exp_{n+1} x} = 0$ by the L'Hopital rule. Thus $\frac{\exp_n(1/\rho)}{\exp_{n+1}(1/\rho)} \approx 0$ implying $2 \exp_n(1/\rho) < \exp_{n+1}(1/\rho)$ for all n. Similarly, $\lim_{x \to \infty} \frac{(\exp_n x)^2}{\exp_{n+1} x} = 0$ by L'Hopital rule implying $\frac{(\exp_n(1/\rho))^2}{\exp_{n+1}(1/\rho)} \approx 0$. Thus $(\exp_n(1/\rho))^2 < \exp_{n+1}(1/\rho)$ for all n. Consequently, $\mathcal{E}_\rho$ is a convex subring of ${}^*\mathbb{C}$ by Lemma 12.2. For its (unique) maximal ideal we have

$$\mathcal{E}_{\rho,0} = \{z \in {}^*\mathbb{C} : \ |z| < \frac{1}{\exp_n(1/\rho)} \ \text{ for all } n \in \mathbb{N}\}.$$

The numbers $e^{1/\rho}$, $\ln \rho$ are both in $\mathcal{E}_\rho \setminus \mathcal{E}_{\rho,0}$ hence, the name **logarithmic-exponential rings** for $\mathcal{E}_\rho$.

12.8 Example (Non-Standard Complex Numbers)**.** The field of the complex numbers $^*\mathbb{C}$ is (trivially) a convex subring of $^*\mathbb{C}$. Its maximal ideal is $\{0\}$.

We observe that

$$(20) \qquad\qquad \mathcal{F} \subset \mathcal{L}_\rho \subset \mathcal{F}_\rho \subset \mathcal{M}_\rho \subset \mathcal{E}_\rho \subset {}^*\mathbb{C},$$

$$(21) \qquad\qquad \{0\} \subset \mathcal{E}_{\rho,0} \subset \mathcal{N}_\rho \subset \mathcal{I}_\rho \subset \mathcal{L}_{\rho,0} \subset \mathcal{I}.$$

13 Spilling Principles

In this section we present several **spilling principles** for a given convex subring $\mathcal{M}$ of $^*\mathbb{C}$ and its maximal ideal (Section 11). These principles generalize the more familiar **underflow and overflow principles** in non-standard analysis (Corollary 13.2). Also in Corollary **??** we show that our spilling principles reduce to the Forth, Fifth and Sixth Principle of Permanence due to Lightstone&Robinson ([57], p. 97-99) in the particular case $\mathcal{M} = \mathcal{M}_\rho$ (Example 12.6).

Let X and Y be two subsets of $^*\mathbb{C}$. We say that X *contains arbitrarily large numbers* in Y if $X \cap Y \neq \varnothing$ and $(\forall z \in X \cap Y)(\exists \zeta \in X \cap Y)(|z| < |\zeta|)$. Similarly, we say that X *contains arbitrarily small numbers* in Y if $X \cap Y \neq \varnothing$ and $(\forall z \in X \cap Y)(\exists \zeta \in X \cap Y)(|z| > |\zeta|)$. With this in mind we have the following result.

13.1 Theorem (Spilling Principles)**.** *Let $\mathcal{M}$ be a convex subring of $^*\mathbb{C}$ (Section 11) and $\mathcal{A} \subseteq {}^*\mathbb{C}$ be an internal set. Then:*

(i) **Overflow of $\mathcal{M}$:** *If $\mathcal{A}$ contains arbitrarily large numbers in $\mathcal{M}$, then $\mathcal{A}$ contains arbitrarily small numbers in $^*\mathbb{C} \setminus \mathcal{M}$. In particular,*

$$\mathcal{M} \setminus \mathcal{M}_0 \subset \mathcal{A} \Rightarrow \mathcal{A} \cap ({}^*\mathbb{C} \setminus \mathcal{M}) \neq \varnothing.$$

(ii) **Underflow of $\mathcal{M} \setminus \mathcal{M}_0$:** *If $\mathcal{A}$ contains arbitrarily small numbers in $\mathcal{M} \setminus \mathcal{M}_0$, then $\mathcal{A}$ contains arbitrarily large numbers in $\mathcal{M}_0$. In particular,*

$$\mathcal{M} \setminus \mathcal{M}_0 \subset \mathcal{A} \Rightarrow \mathcal{A} \cap \mathcal{M}_0 \neq \varnothing.$$

(iii) **Overflow of $\mathcal{M}_0$:** *If $\mathcal{A}$ contains arbitrarily large numbers in $\mathcal{M}_0$, then $\mathcal{A}$ contains arbitrarily small numbers in $\mathcal{M} \setminus \mathcal{M}_0$. In particular,*

$$\mathcal{M}_0 \subset \mathcal{A} \Rightarrow \mathcal{A} \cap (\mathcal{M} \setminus \mathcal{M}_0) \neq \varnothing.$$

(iv) Underflow of $^*\mathbb{C} \setminus \mathcal{M}$: *If $\mathcal{A}$ contains arbitrarily small numbers in $^*\mathbb{C}\setminus\mathcal{M}$, then $\mathcal{A}$ contains arbitrarily large numbers in $\mathcal{M}$. In particular,*

$$^*\mathbb{C} \setminus \mathcal{M} \subset \mathcal{A} \Rightarrow \mathcal{A} \cap (\mathcal{M} \setminus \mathcal{M}_0) \neq \varnothing.$$

Proof. (i) If $\mathcal{A}$ is unbounded in $^*\mathbb{C}$, there is nothing to prove. If $\mathcal{A}$ is bounded in $^*\mathbb{C}$, then $\sup(|\mathcal{A}|) = x$ exists in $^*\mathbb{R}$, where $|\mathcal{A}| = \{|z| : z \in \mathcal{A}\}$. Notice that $x \notin \mathcal{M}$ because $x \in \mathcal{M}$ contradicts the assumption for $\mathcal{A}$. Next, there exists $z \in \mathcal{A}$ such that $x/2 < |z| < x$ by the choice of x and we have $z \notin \mathcal{M}$ because $x/2 \notin \mathcal{M}$ (notice that $x/2 \in \mathcal{M}$ implies $x/2+x/2 \in \mathcal{M}$). We just proved that $\mathcal{A}\cap(^*\mathbb{C}\setminus\mathcal{M}) \neq \varnothing$. It remains to show that $\mathcal{A}\cap(^*\mathbb{C}\setminus\mathcal{M})$ does not have a lower bound in $^*\mathbb{C} \setminus \mathcal{M}$. Suppose (on the contrary) that there exists $\lambda \in {}^*\mathbb{C} \setminus \mathcal{M}$ such that $\lambda \leq |z|$ for all $z \in \mathcal{A}\cap(^*\mathbb{C} \setminus \mathcal{M})$. The set $\mathcal{A}_\lambda = \{z \in \mathcal{A} : |z| < \lambda\}$ is internal and we have $\mathcal{A}_\lambda = \mathcal{A} \cap \mathcal{M}$ by the choice of λ. It follows that $\mathcal{A}_\lambda$ has arbitrarily large elements in $\mathcal{M}$ because $\mathcal{A}$ has arbitrarily large elements in $\mathcal{M}$ by assumption. We conclude that $\mathcal{A}_\lambda \cap (^*\mathbb{C} \setminus \mathcal{M}) \neq \varnothing$ by what was proved above. Thus there exists $z \in \mathcal{A} \cap (^*\mathbb{C} \setminus \mathcal{M})$ such that $|z| < \lambda$, a contradiction.

(ii) follows immediately from (i) and the fact that $z \in \mathcal{M} \setminus \mathcal{M}_0$ implies $1/z \in \mathcal{M} \setminus \mathcal{M}_0$ and also that $z \in {}^*\mathbb{C} \setminus \mathcal{M}$ implies $1/z \in \mathcal{M}_0$ by part (i) of Lemma 11.3.

The proof of (iii) is similar to the proof of (i) and we leave it to the reader.

(iv) follows immediately from (iii) and the fact that $z \in \mathcal{M}_0 \setminus \{0\}$ implies $1/z \in {}^*\mathbb{C} \setminus \mathcal{M}$ and also that $z \in \mathcal{M} \setminus \mathcal{M}_0$ implies $1/z \in \mathcal{M} \setminus \mathcal{M}_0$. $\square$

Here are the more familiar spilling (underflow and overflow) principles about $\mathcal{F}(^*\mathbb{C})$, $\mathcal{I}(^*\mathbb{C})$ and $\mathcal{L}(^*\mathbb{C})$.

13.2 Corollary (The Usual Spilling Principles). *Let $\mathcal{A} \subseteq {}^*\mathbb{C}$ be an internal set. Then:*

(i) Overflow of $\mathcal{F}(^*\mathbb{C})$: *If $\mathcal{A}$ contains arbitrarily large finite numbers, then $\mathcal{A}$ contains arbitrarily small infinitely large numbers. In particular,*

$$\mathcal{F}(^*\mathbb{C}) \setminus \mathcal{I}(^*\mathbb{C}) \subset \mathcal{A} \Rightarrow \mathcal{A} \cap \mathcal{L}(^*\mathbb{C}) \neq \varnothing.$$

(ii) Underflow of $\mathcal{F}(^*\mathbb{C}) \setminus \mathcal{I}(^*\mathbb{C})$: *If $\mathcal{A}$ contains arbitrarily small finite non-infinitesimals, then $\mathcal{A}$ contains arbitrarily large infinitesimals. In particular,*

$$\mathcal{F}(^*\mathbb{C}) \setminus \mathcal{I}(^*\mathbb{C}) \subset \mathcal{A} \Rightarrow \mathcal{A} \cap \mathcal{I}(^*\mathbb{C}) \neq \varnothing.$$

(iii) Overflow of $\mathcal{I}(^*\mathbb{C})$: *If $\mathcal{A}$ contains arbitrarily large infinitesimals, then $\mathcal{A}$ contains arbitrarily small finite non-infinitesimals. In particular,*

$$\mathcal{I}(^*\mathbb{C}) \subset \mathcal{A} \Rightarrow \mathcal{A} \cap (\mathcal{F}(^*\mathbb{C}) \setminus \mathcal{I}(^*\mathbb{C})) \neq \varnothing.$$

(iv) Underflow of $\mathcal{L}(^*\mathbb{C})$: *If $\mathcal{A}$ contains arbitrarily small infinitely large numbers, then $\mathcal{A}$ contains arbitrarily large finite numbers. In particular,*

$$\mathcal{L}(^*\mathbb{C}) \subset \mathcal{A} \Rightarrow \mathcal{A} \cap (\mathcal{F}(^*\mathbb{C}) \setminus \mathcal{I}(^*\mathbb{C})) \neq \varnothing.$$

Proof. The result follows directly from the previous theorem in the particular case of $\mathcal{M} = \mathcal{F}(^*\mathbb{C})$ taking into account that in this case $\mathcal{M}_0 = \mathcal{I}(^*\mathbb{C})$ (Example 12.3). ∎

13.3 Corollary (Generating Sequences). **(i)** *Let (δ_n) be a decreasing generating sequence in $^*\mathbb{R}$ and $\mathcal{M}$ be the convex subring of $^*\mathbb{C}$ generated by (δ_n) (part (i) of Lemma 12.2). Let $(^*\delta_n)$ be the non-standard extension of (δ_n). Then*

(a) $z \in \mathcal{M}$ iff $(\exists \nu \in {}^*\mathbb{N} \setminus \mathbb{N})(|z| \leq {}^*\delta_\nu)$.

(b) $z \in \mathcal{M}_0$ iff $(\forall \nu \in {}^*\mathbb{N} \setminus \mathbb{N})(|z| < 1/{}^*\delta_\nu)$.

(ii) *Let (λ_n) be a increasing generating sequence in $^*\mathbb{R}$ and $\mathcal{M}$ be the convex subring of $^*\mathbb{C}$ generated by (λ_n) (part (ii) of Lemma 12.2). Let $(^*\lambda_n)$ be the non-standard extension of (λ_n). Then*

(a) $z \in \mathcal{M}$ iff $(\forall \nu \in {}^*\mathbb{N} \setminus \mathbb{N})(|z| < {}^*\lambda_\nu)$.

(b) $z \in \mathcal{M}_0$ iff $(\exists \nu \in {}^*\mathbb{N} \setminus \mathbb{N})(|z| \leq 1/{}^*\lambda_\nu)$.

Proof. (i)-(a): Suppose $z \in \mathcal{M}$. The internal set $\mathcal{A} = \{n \in {}^*\mathbb{N} : |z| < {}^*\delta_n\}$ contains $\mathbb{N}$ by assumption hence, there exists $\nu \in ({}^*\mathbb{N} \setminus \mathbb{N}) \cap \mathcal{A}$ (as required) by the overflow of $\mathcal{F}(^*\mathbb{C})$ (Corollary 13.2) since $^*\mathbb{N} \setminus \mathbb{N} \subset \mathcal{F}(^*\mathbb{C})$). Conversely, $|z| < {}^*\delta_\nu$ for some $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$ implies $z \in \mathcal{M}$ by the convexity of $\mathcal{M}$, since $^*\delta_\nu < \delta_n$ for all $n \in \mathbb{N}$.
  (i)-(b) follows immediately from (i)-(a) and part (i) of Lemma 11.3.
  The proof of (ii) is similar to the proof of (i) and leave it to the reader. ∎

In the next corollary we derive the Robinson's Principles of Permanence as a particular case of our more general Spilling Principles for $\mathcal{M} = \mathcal{M}_\rho$

(Example 12.6). We should note that in Lightstone&Robinson ([57], p. 97-99) the numbers in $\mathcal{N}_\rho(^*\mathbb{C})$ are called **iota numbers** and the numbers in $^*\mathbb{C} \setminus \mathcal{M}_\rho$ are called **mega numbers**.

13.4 Corollary (Robinson's Principles of Permanence). *Let* $\mathcal{A} \subseteq {}^*\mathbb{C}$ *be an internal set. Then:*

(i) Overflow of $\mathcal{M}_\rho(^*\mathbb{C})$**:** *If* $\mathcal{A}$ *contains arbitrarily large numbers in* $\mathcal{M}_\rho(^*\mathbb{C})$, *then* $\mathcal{A}$ *contains arbitrarily small numbers in* $^*\mathbb{C} \setminus \mathcal{M}_\rho(^*\mathbb{C})$. *In particular,*
$$\mathcal{M}_\rho(^*\mathbb{C}) \setminus \mathcal{N}_\rho(^*\mathbb{C}) \subset \mathcal{A} \Rightarrow \mathcal{A} \cap (^*\mathbb{C} \setminus \mathcal{M}_\rho(^*\mathbb{C})) \neq \varnothing.$$

(ii) Underflow of $\mathcal{M}_\rho(^*\mathbb{C}) \setminus \mathcal{N}_\rho(^*\mathbb{C})$**:** *If* $\mathcal{A}$ *contains arbitrarily small numbers in* $\mathcal{M}_\rho(^*\mathbb{C}) \setminus \mathcal{N}_\rho(^*\mathbb{C})$, *then* $\mathcal{A}$ *contains arbitrarily large numbers in* $\mathcal{N}_\rho(^*\mathbb{C})$. *In particular,*

$$\mathcal{M}_\rho(^*\mathbb{C}) \setminus \mathcal{N}_\rho(^*\mathbb{C}) \subset \mathcal{A} \Rightarrow \mathcal{A} \cap \mathcal{N}_\rho(^*\mathbb{C}) \neq \varnothing.$$

(iii) Overflow of $\mathcal{N}_\rho(^*\mathbb{C})$**:** *If* $\mathcal{A}$ *contains arbitrarily large numbers in* $\mathcal{N}_\rho(^*\mathbb{C})$, *then* $\mathcal{A}$ *contains arbitrarily small numbers in* $\mathcal{M}_\rho(^*\mathbb{C}) \setminus \mathcal{N}_\rho(^*\mathbb{C})$. *In particular,*

$$\mathcal{N}_\rho(^*\mathbb{C}) \subset \mathcal{A} \Rightarrow \mathcal{A} \cap (\mathcal{M}_\rho(^*\mathbb{C}) \setminus \mathcal{N}_\rho(^*\mathbb{C})) \neq \varnothing.$$

(iv) Underflow of $^*\mathbb{C} \setminus \mathcal{M}_\rho(^*\mathbb{C})$**:** *If* $\mathcal{A}$ *contains arbitrarily small numbers in* $^*\mathbb{C} \setminus \mathcal{M}_\rho(^*\mathbb{C})$, *then* $\mathcal{A}$ *contains arbitrarily large numbers in* $\mathcal{M}_\rho(^*\mathbb{C})$. *In particular,*

$$^*\mathbb{C} \setminus \mathcal{M}_\rho(^*\mathbb{C}) \subset \mathcal{A} \Rightarrow \mathcal{A} \cap (\mathcal{M}_\rho(^*\mathbb{C}) \setminus \mathcal{N}_\rho(^*\mathbb{C})) \neq \varnothing.$$

Proof. These results follow immediately from our general Spilling Principles (Theorem 13.1) for $\mathcal{M} = \mathcal{M}_\rho$ (Example 12.6). $\square$

14   Asymptotic Fields

In this section we study particular type of algebraically closed subfields of $^*\mathbb{C}$ called *asymptotic fields*.

14.1 Definition (Asymptotic Fields).   1. An **asymptotic field** is a field of the form $\mathcal{M}/\mathcal{M}_0$, where $\mathcal{M}$ is a convex subring of $^*\mathbb{C}$ and $\mathcal{M}_0$ is its maximal ideal (Definition 11.1). A field which is isomorphic to a field of the form $\mathcal{M}/\mathcal{M}_0$ will be also called *asymptotic*. We shall call the elements of $\mathcal{M}/\mathcal{M}_0$ **complex $\mathcal{M}$-asymptotic numbers** (or simply *asymptotic numbers* if no confusion could arise). We denote by $q_\mathcal{M} : \mathcal{M} \to \mathcal{M}/\mathcal{M}_0$ the corresponding **quotient mapping**.

2. The elements of $q_\mathcal{M}[^*\mathbb{R} \cap \mathcal{M}]$ are called **real $\mathcal{M}$-asymptotic numbers** (or simply *real asymptotic numbers* if no confusion could arise). We shall sometime refer to $q_\mathcal{M}[^*\mathbb{R} \cap \mathcal{M}]$ as the **real part** of $\mathcal{M}/\mathcal{M}_0$ and denote it by $\Re e(\mathcal{M}/\mathcal{M}_0)$.

3. We define the **order relation** in $q_\mathcal{M}[^*\mathbb{R} \cap \mathcal{M}]$ by: $q_\mathcal{M}(x) > 0$ if $q_\mathcal{M}(x) \neq 0$ and $x > 0$ in $^*\mathbb{R}$.

4. We define the **absolute value** $|\cdot| : \mathcal{M}/\mathcal{M}_0 \to q_\mathcal{M}[^*\mathbb{R} \cap \mathcal{M}]$ by the formula $|q_\mathcal{M}(z)| = q_\mathcal{M}(|z|)$.

14.2 Notation (Suppressing $\mathcal{M}$). Let $\mathcal{M}$ be a convex subring of $^*\mathbb{C}$ (Definition 14.1). We shall often **suppress the dependence on $\mathcal{M}$** and use the following simplified notation:

(i) If $z \in \mathcal{M}$, we shall often write $\widehat{z}$ instead of the more precise $q_\mathcal{M}(z)$. Also, we shall write $z \to \widehat{z}$ for the quotient mapping $q_\mathcal{M}$.

(ii) If $S \subseteq {^*\mathbb{C}}$, we denote $\widehat{S} = q_\mathcal{M}[S \cap \mathcal{M}]$. Observe that we have $\widehat{^*\mathbb{C}} = \widehat{\mathcal{M}} = q_\mathcal{M}[\mathcal{M}]$. Also, we have $\widehat{^*\mathbb{R}} = q_\mathcal{M}[^*\mathbb{R} \cap \mathcal{M}]$. We shall often prefer the simpler notation $\widehat{^*\mathbb{C}}$ instead of the more precise $\mathcal{M}/\mathcal{M}_0$ or $\widehat{\mathcal{M}}$, when no confusion could arise. Also, we shall often write $\widehat{^*\mathbb{R}}$ instead of $q_\mathcal{M}[^*\mathbb{R} \cap \mathcal{M}]$. Summarizing, we have

$$(22) \quad \widehat{^*\mathbb{C}} = \widehat{\mathcal{M}} = q_\mathcal{M}[\mathcal{M}] \quad \text{and} \quad \widehat{^*\mathbb{R}} = \Re e(\widehat{^*\mathbb{C}}) = \Re e(\widehat{\mathcal{M}}) = q_\mathcal{M}[^*\mathbb{R} \cap \mathcal{M}].$$

(iii) In this notation the **order relation** in $\widehat{^*\mathbb{R}}$ (defined above) is phrase as follows: $\widehat{x} > 0$ in $\widehat{^*\mathbb{R}}$ if $\widehat{x} \neq 0$ and $x > 0$ in $^*\mathbb{R}$.

(iv) In this notation the **absolute value** $|\cdot| : \widehat{^*\mathbb{C}} \to \widehat{^*\mathbb{R}}$ (defined above) is given by the formula $|\widehat{z}| = \widehat{|z|}$.

(v) If $S \subseteq \mathbb{C}$, we shall often write simply S instead of the more precise $q_{\mathcal{M}}[S \cap \mathcal{M}]$ or $\widehat{S}$. In particular, we shall often write simply $\mathbb{C}$ instead of $\widehat{\mathbb{C}}$ or $q_{\mathcal{M}}[\mathbb{C} \cap \mathcal{M}]$. Similarly, we shall often write simply $\mathbb{R}$ instead of $\widehat{\mathbb{R}}$ or $q_{\mathcal{M}}[\mathbb{R} \cap \mathcal{M}]$.

(vi) When we dealing not with one (fixed) but rather with two (or more than two) convex subrings, say $\mathcal{M}_1$ and $\mathcal{M}_2$, we shall prefer the original notation introduced in Definition 14.1 or the "hat" notation $\widehat{\mathcal{M}_1}$ and $\widehat{\mathcal{M}_2}$ instead of $\widehat{^*\mathbb{C}}$.

14.3 Theorem (Asymptotic Fields). *Let $\mathcal{M}$ be a convex subring in $^*\mathbb{C}$ and $\widehat{^*\mathbb{C}} = \mathcal{M}/\mathcal{M}_0$ be the corresponding asymptotic field and let $\widehat{^*\mathbb{R}} = q_{\mathcal{M}}[^*\mathbb{R} \cap \mathcal{M}]$ be its real part. Then:*

(i) *$\widehat{^*\mathbb{C}}$ is a field. Also, $\widehat{^*\mathbb{R}}$ is a totally ordered field and we have $\widehat{^*\mathbb{C}} = \widehat{^*\mathbb{R}}(i)$.*

(iv) *Either of $\widehat{^*\mathbb{C}}$ or $\widehat{^*\mathbb{R}}$ is an archimedean field iff $\mathcal{M} = \mathcal{F}(^*\mathbb{C})$.*

Proof. (i) $\widehat{^*\mathbb{C}}$ is a field because $\mathcal{M}_0$ is a maximal ideal in $\mathcal{M}$ by Lemma 11.3. $\widehat{^*\mathbb{R}}$ is a real field because $^*\mathbb{R} \cap \mathcal{M}$ is a real ring and $^*\mathbb{R} \cap \mathcal{M}_0$ is a convex maximal ideal in $^*\mathbb{R} \cap \mathcal{M}$ by Lemma 11.3. The connection $\widehat{^*\mathbb{C}} = \widehat{^*\mathbb{R}}(i)$ follows from $\mathcal{M} = (^*\mathbb{R} \cap \mathcal{M})(i)$ (Lemma 11.2).

(ii) Either $\widehat{^*\mathbb{C}}$ or $\widehat{^*\mathbb{C}}$ is an archimedean field iff $\mathcal{M}$ is an archimedean ring iff $\mathcal{M} = \mathcal{F}(^*\mathbb{C})$ by Lemma 11.2. $\qquad\square$

Our next goal is to show that every asymptotic field $\widehat{^*\mathbb{C}}$ is algebraically closed field and its real part $\widehat{^*\mathbb{R}}$ is a real closed field. We start with the following observation.

14.4 Lemma (Isomorphic Fields). *Let $\mathcal{M}$ be (as before) a convex subring in $^*\mathbb{C}$. Let $\mathbb{K}$ be a field which is a subring of $\mathcal{M}$ and let $\widehat{\mathbb{K}} = q_{\mathcal{M}}[\mathbb{K}]$ (# 1 of Definition 14.1). Then*

(i) *The fields $\mathbb{K}$ and $\widehat{\mathbb{K}}$ are isomorphic under the mapping $z \to \widehat{z}$ (or, equivalently, under the quotient mapping $q_{\mathcal{M}}|\mathbb{K}$). In particular, $\mathbb{C}$ and $\widehat{\mathbb{C}}$ are isomorphic and $\mathbb{R}$ and $\widehat{\mathbb{R}}$ are isomorphic.*

(ii) *The following are equivalent:* $\mathcal{F}(^*\mathbb{C}) \subseteq \mathbb{K}$ iff $\mathcal{I}(^*\mathbb{C}) \subseteq \mathbb{K}$ iff $\mathcal{L}(^*\mathbb{C}) \subseteq \mathbb{K}$ iff $\mathbb{K} = {}^*\mathbb{C}$.

(iii) *Similarly, the following are equivalent:* $\mathcal{F}(^*\mathbb{R}) \subseteq \mathbb{K}$ iff $\mathcal{I}(^*\mathbb{R}) \subseteq \mathbb{K}$ iff $\mathcal{L}(^*\mathbb{R}) \subseteq \mathbb{K}$ iff $\mathbb{K} = {}^*\mathbb{R}$.

Proof. (i) We observe that $\mathbb{K} \cap \mathcal{M}_0 = \{0\}$. Indeed, suppose (on the contrary) that $z \in \mathbb{K} \cap \mathcal{M}_0$ for some $z \neq 0$. It follows $1/z \in \mathbb{K}$ (since $\mathbb{K}$ is a field) and also $1/z \notin \mathcal{M}$ (by the definition of $\mathcal{M}_0$) which contradicts the assumption $\mathbb{K} \subseteq \mathcal{M}$. Consequently, $\mathbb{K}$ and $\widehat{\mathbb{K}}$ are isomorphic. We leave the verification of (ii) and (iii) to the reader. $\square$

The notation introduced in part (v) of Notation 14.2 is justified by the following result.

14.5 Corollary (Embedding of Complex Numbers). *The mapping* $\sigma : \mathbb{C} \to \widehat{^*\mathbb{C}}$*, defined by* $\sigma(z) = \widehat{z}$*, is a **field embedding** of* $\mathbb{C}$ *into* $\widehat{^*\mathbb{C}}$*.*

Proof. The result follows from the above lemma since $\mathbb{C} \subseteq \mathcal{M}$ by Lemma 11.2. $\square$

14.6 Definition (Maximal Fields). Let $\mathcal{M}$ be a subring of $^*\mathbb{C}$ containing $\mathbb{C}$. A subfield $\mathbb{M}$ of $^*\mathbb{C}$ is called **maximal in** $\mathcal{M}$ if: (a) $\mathbb{M}$ is a subring of $\mathcal{M}$ and $\mathbb{M}$ also contains a copy of $\mathbb{C}$, i.e. $\mathbb{C} \subseteq \mathbb{M} \subseteq \mathcal{M}$; (b) There is no subfield $\mathbb{K}$ of $^*\mathbb{C}$ which is a subring of $\mathcal{M}$ and which is a proper field extension of $\mathbb{M}$. We denote by $\mathcal{M}ax(\mathcal{M})$ the **set of all maximal fields** in $\mathcal{M}$.

For example, the field of the complex numbers $\mathbb{C}$ is a maximal field in the ring of finite numbers $\mathcal{F}(^*\mathbb{C})$ (Example 12.3).

14.7 Lemma (Existence of Maximal Fields). *Let* $\mathbb{K}$ *be subfield of* $^*\mathbb{C}$ *such that* $\mathbb{C} \subseteq \mathbb{K} \subseteq \mathcal{M}$*. Then there exists a maximal field* $\mathbb{M} \in \mathcal{M}ax(\mathcal{M})$ *which is a field extension of* $\mathbb{K}$*. Consequently,* $\mathcal{M}ax(\mathcal{M}) \neq \varnothing$*.*

Proof. Let $\mathcal{L}_{\mathbb{K}}$ denote the set of all subfields $\mathbb{L}$ of $^*\mathbb{C}$ such that $\mathbb{K} \subseteq \mathbb{L} \subseteq \mathcal{M}$. We order $\mathcal{L}_{\mathbb{K}}$ by inclusion. We obviously have $\mathcal{L}_{\mathbb{K}} \neq \varnothing$, since $\mathbb{K} \in \mathcal{L}_{\mathbb{K}}$. Also, we observe that if S is a totally ordered subset of $\mathcal{L}_{\mathbb{K}}$ under the inclusion $\subset$, then $\bigcup_{\mathbb{L} \in S} \mathbb{L} \in \mathcal{L}_{\mathbb{K}}$. Thus $\mathcal{L}_{\mathbb{K}}$ has a maximal element, say $\mathbb{M}$, as required, by Zorn's lemma. Consequently, $\mathcal{M}ax(\mathcal{M}) \neq \varnothing$ because $\mathbb{C}$ is a subring of $\mathcal{M}$ by Lemma 11.2. $\square$

14.8 Theorem (Field of Representatives). *Let $\mathcal{M}$ be (as before) a convex subring of $^*\mathbb{C}$ and $\mathbb{M} \in \mathcal{M}ax(\mathcal{M})$. Then:*

(i) $\mathbb{M}$ *and* $\widehat{\mathbb{M}}$ *are* **algebraically closed isomorphic fields**.

(ii) *Let $z_0 \in \mathcal{M}$ be a point which is* **away from** $\mathbb{M}$ *in the sense that*

$$(23) \qquad (\forall z \in \mathbb{M})\,(z - z_0 \notin \mathcal{M}_0).$$

Then $P(z_0) \notin \mathcal{M}_0$ for any non-zero polynomials P with coefficients in $\mathbb{M}$. Consequently, the field of the rational functions $\mathbb{M}(z_0)$ is a proper field extension of $\mathbb{M}$ within $\mathcal{M}$, in symbol, $\mathbb{M} \subsetneqq \mathbb{M}(z_0) \subseteq \mathcal{M}$.

(iii) *We have the following* **characterization** *of $\mathcal{M}$ and $\mathcal{M}_0$*

$$
\begin{aligned}
(24) \qquad & \mathcal{M} = \{z \in {}^*\mathbb{C} \mid (\exists \varepsilon \in \mathbb{M}_+)(|z| \leq \varepsilon\}, \\
(25) \qquad & \mathcal{M}_0 = \{z \in {}^*\mathbb{C} \mid (\forall \varepsilon \in \mathbb{M}_+)(|z| < \varepsilon\},
\end{aligned}
$$

where $\mathbb{M}_+ = \{|z| : z \in \mathbb{M}, z \neq 0\}$.

(iv) *We have $\mathcal{M} = \mathbb{M} \oplus \mathcal{M}_0$ in the sense that every $z \in \mathcal{M}$ has a unique asymptotic expansion $z = c + dz$, where $c \in \mathbb{M}$ and $dz \in \mathcal{M}_0$. Consequently, $\mathbb{M}$ is a* **field of representatives** *for $\widehat{{}^*\mathbb{C}}$ in the sense that $\widehat{{}^*\mathbb{C}} = \widehat{\mathbb{M}}$ or $\widehat{\mathcal{M}} = \widehat{\mathbb{M}}$ depending on the choice of the notation (Notation 14.2).*

Proof. (i) We intend to show that $\mathbb{M}$ is algebraically closed. We denote by $\mathrm{cl}(\mathbb{M})$ the relative algebraic closure of $\mathbb{M}$ in $^*\mathbb{C}$. Since $^*\mathbb{C}$ is an algebraically closed field, so is $\mathrm{cl}(\mathbb{M})$. To show that $\mathrm{cl}(\mathbb{M}) \subseteq \mathcal{M}$, suppose that $z \in \mathrm{cl}(\mathbb{M})$. Since z is algebraic over $\mathbb{M}$, it follows that z is a root of some polynomial $P(x) = x^n + a_1\,x^{n-1} + \cdots + a_n$ with coefficients in $\mathbb{M}$. The estimation $|z| \leq 1 + |a_1| + \cdots + |a_n|$ implies $z \in \mathcal{M}$ by the convexity of $\mathcal{M}$. Now, $\mathbb{M} \subseteq \mathrm{cl}(\mathbb{M}) \subseteq \mathcal{M}$ implies $\mathbb{M} = \mathrm{cl}(\mathbb{M})$ by the maximality of $\mathbb{M}$ in $\mathcal{M}$. The fields $\mathbb{M}$ and $\widehat{\mathbb{M}}$ are isomorphic by Lemma 14.4.

(ii) Suppose (on the contrary) that $P(z_0) \in \mathcal{M}_0$ for some polynomial P. It follows that $\widehat{P(z_0)} = 0$ implying $\widehat{P}(\widehat{z_0}) = 0$, where $\widehat{P}$ denotes the polynomial, obtained from P by replacing the coefficients a_k in P by $\widehat{a_k}$. Since $\widehat{\mathbb{M}}$ is an algebraically closed field, it follows that $\widehat{z_0} \in \widehat{\mathbb{M}}$ meaning $z_0 - z \in \mathcal{M}_0$ for some $z \in \mathbb{M}$, a contradiction.

(iii) Let $z_0 \in \mathcal{M}$ and suppose (on the contrary) that $(\forall \varepsilon \in \mathbb{M}_+)(|z_0| > \varepsilon)$. Observe that z_0 is away from $\mathbb{M}$ in the sense of (23). Thus $\mathbb{M}(z_0)$ is a proper field extension of $\mathbb{M}$ within $\mathcal{M}$ by (ii), contradicting the maximality of $\mathbb{M}$. This proves the formula (24) about $\mathcal{M}$ since the inclusion in the opposite direction follows from the convexity of $\mathcal{M}$. Let $z \in \mathcal{M}_0$. If $z = 0$, there is nothing to prove. If $z \neq 0$, we have $1/z \notin \mathcal{M}$ by the definition of $\mathcal{M}_0$. Next, suppose (on the contrary) that $|z| \geq \varepsilon$ for some $\varepsilon \in \mathbb{M}_+$. It follows that $|1/z| \leq 1/\varepsilon$ implying $1/z \in \mathcal{M}$ by formula (24), a contradiction. Conversely, suppose that $|z| < \varepsilon$ for all $\varepsilon \in \mathbb{M}_+$ and some $z \in {}^*\mathbb{C}$. It follows that $1/\varepsilon < |1/z|$ for all $\varepsilon \in \mathbb{M}_+$ implying $1/z \notin \mathcal{M}$ by the formula (24). It follows $z \in \mathcal{M}_0$ (by part (i) of Lemma 11.3), which proves formula (25).

(iv) To show the existence of asymptotic expansion, suppose (on the contrary) that there exists $z \in \mathcal{M}$ such that $z - c \notin \mathcal{M}_0$ for all $c \in \mathbb{M}$. We have $\mathbb{M} \subsetneq \mathbb{M}(z) \subseteq M$ by (ii), contradicting the maximality of $\mathbb{M}$. To show the uniqueness, suppose that $c + dz = c_1 + dz_1$. It follows $c - c_1 \in \mathcal{M}_0$ thus $c - c_1 = 0$ (as required) since $\mathbb{M} \cap \mathcal{M}_0 = \{0\}$. $\square$

14.9 Definition (M-Standard Part Mapping). The mapping $\mathrm{st}_\mathbb{M} : \mathcal{M} \to {}^*\mathbb{C}$, defined by $\mathrm{st}_\mathbb{M}(c + dz) = c$, is called $\mathbb{M}$-**standard part mapping**.

14.10 Lemma (M-Standard Part Mapping). **(i)** *For every $z \in \mathcal{M}$ we have*
$z = \mathrm{st}_\mathbb{M}(z) + dz$, *where $\mathrm{st}_\mathbb{M}(z) \in \mathbb{M}$ and $dz \in \mathcal{M}_0$.*

(ii) *The $\mathbb{M}$-standard part mapping $\mathrm{st}_\mathbb{M} : \mathcal{M} \to {}^*\mathbb{C}$ is a ring homomorphism with range $\mathrm{st}_\mathbb{M}[\mathcal{M}] = \mathbb{M}$.*

(iii) $\mathrm{st}_\mathbb{M}$ *is an extension of the usual standard part mapping $\mathrm{st} : \mathcal{F}({}^*\mathbb{C}) \to \mathbb{C}$, i.e. $\mathrm{st}_\mathbb{M} \,|\, \mathcal{F}({}^*\mathbb{C}) = \mathrm{st}$.*

Proof. (i) is a notational modification of the result of part (iv) of Theorem 14.8.

(ii) The fact that $\mathrm{st}_\mathbb{M}$ is homomorphism follows directly from the formula $z = \mathrm{st}_\mathbb{M}(z) + dz$.

(iii) follows directly from the fact that $\mathbb{C} \subseteq \mathbb{M}$ and $\mathcal{M}_0 \subseteq \mathcal{I}({}^*\mathbb{C})$. $\square$

14.11 Theorem (Algebraically Closed Field). *Let $\mathcal{M}$ be a convex subring of ${}^*\mathbb{C}$ and let $\widehat{{}^*\mathbb{C}} = \widehat{\mathcal{M}}$ be its asymptotic field (Notation 14.2). Let $\mathbb{M} \in Max(\mathcal{M})$. Then the fields $\mathbb{M}$ and $\widehat{{}^*\mathbb{C}} = \widehat{\mathcal{M}}$ are isomorphic under the mapping*

$z \to \widehat{z}$ (or, equivalently, under the quotient mapping $q_{\mathcal{M}} \,|\, \mathbb{M}$). The situation can be summarized in the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{q_{\mathcal{M}}} & \widehat{^*\mathbb{C}} \\[2pt]
id \uparrow & & \uparrow id \\[2pt]
\mathbb{M} & \xrightarrow{q_{\mathcal{M}}|\mathbb{M}} & \widehat{\mathbb{M}}.
\end{array}
$$

Consequently, every asymptotic field is an **algebraically closed field** and the real part of an asymptotic field is a **real closed field**.

Proof. The fields $\mathbb{M}$ and $\widehat{\mathbb{M}}$ are isomorphic by Lemma 14.4. Also, $\widehat{^*\mathbb{C}}$ and $\widehat{\mathbb{M}}$ are isomorphic (under the identity) since $\widehat{^*\mathbb{C}} = \widehat{\mathbb{M}}$ by Theorem 14.8. Thus $\widehat{^*\mathbb{C}}$ is an algebraically closed field because $\mathbb{M}$ is an algebraically closed field (Theorem 14.8). The field $\widehat{^*\mathbb{R}}$ is a real closed by Artin-Schreier theorem (Marker, Messmer, Pillay [62], p. 9), since $\widehat{^*\mathbb{C}} = \widehat{^*\mathbb{R}}(i)$ by Theorem 14.3. $\square$

14.12 Definition (Convex Cover). Let S be a subset of $^*\mathbb{C}$. The set $\mathrm{cov}(S) = \{z \in {^*\mathbb{C}} : |z| \leq |\zeta| \text{ for some } \zeta \in S\}$ is the **convex cover** of S in $^*\mathbb{C}$.

14.13 Lemma. *Let S be a subring of $^*\mathbb{C}$ which is closed under the absolute value in the sense that $z \in S$ implies $|z| \in S$. Then $\mathrm{cov}(S)$ is a convex subring of $^*\mathbb{C}$ (Definition 11.1). In particular, $\mathrm{cov}(\mathbb{K})$ is a convex subring of $^*\mathbb{C}$ for any algebraically closed subfield $\mathbb{K}$ of $^*\mathbb{C}$. Also $\mathrm{cov}(\mathcal{M}) = \mathcal{M}$ for any convex subring $\mathcal{M}$ of $^*\mathbb{C}$.*

Proof. We leave the details to the reader. $\square$

14.14 Theorem (A Characterization). *Let $\mathbb{K}$ be a subring of $^*\mathbb{C}$ which is closed under the absolute value. Then the following are equivalent:*

(i) *$\mathbb{K}$ is an asymptotic field (Definition 14.1).*

(ii) *$\mathbb{K}$ is a maximal field in $\mathrm{cov}(\mathbb{K})$, in symbol, $\mathbb{K} \in \mathcal{M}ax(\mathrm{cov}(\mathbb{K}))$ (Definition 14.6).*

Proof. (i)$\Rightarrow$(ii) Suppose that $\mathbb{K}$ is isomorphic to some asymptotic field $\widehat{\mathcal{M}}$, where $\mathcal{M}$ is a convex subring of $^*\mathbb{C}$. Suppose (on the contrary) that $\mathbb{K} \notin \mathcal{M}ax(\mathrm{cov}(\mathbb{K}))$. It follows that $\mathbb{K} \notin \mathcal{M}ax(\mathcal{M})$ because $\mathrm{cov}(\mathbb{K}) \subseteq \mathcal{M}$. Thus

there exists a maximal field $\mathbb{M} \in \mathcal{M}ax(\mathcal{M})$ which is a proper field extension of $\mathbb{K}$ (Lemma 14.7). On the other hand, $\widehat{\mathcal{M}}$ and $\mathbb{M}$ are isomorphic by Theorem 14.11 thus $\mathbb{K}$ and $\mathbb{M}$ must be isomorphic, a contradiction.

(i)$\Leftarrow$(ii) $\mathbb{K} \in \mathcal{M}ax(\text{cov}(\mathbb{K}))$ implies that the fields $\mathbb{K}$ and $\widehat{\text{cov}(\mathbb{K})}$ are isomorphic by Theorem 14.11. Thus $\mathbb{K}$ is an asymptotic field since $\text{cov}(\mathbb{K})$ is a convex subrings of $^*\mathbb{C}$. $\square$

14.15 Example. Let $\mathbb{C}(t)$ be the field of rational functions in one variable with complex coefficients. Let ρ be a positive infinitesimal in $^*\mathbb{R}$. Then $\mathbb{C}(\rho)$ is a subfield of $^*\mathbb{C}$ which is closed under the absolute value. It is easy to see that $\text{cov}(\mathbb{C}(\rho)) = \mathcal{M}_\rho$ (Example 12.6). Thus $\mathbb{C}(\rho)$ is not an asymptotic field because $\mathbb{C}(\rho)$ is not maximal in $\mathcal{M}_\rho$. Indeed, we have $\mathbb{C}(\rho) \subsetneqq \mathbb{C}(\rho^{\mathbb{Z}}) \subset {}^\rho\mathbb{C} \subset \mathcal{M}_\rho$, where $\mathbb{C}(\rho^{\mathbb{Z}})$ stands for the field of the Laurent series with complex coefficients. Similarly, the field of the Levi-Civita series $\mathbb{C}\langle\rho\rangle$ and the Hanh field $\mathbb{C}(\rho^{\mathbb{R}})$ are not asymptotic (Section 18).

15 Embeddings in $^*\mathbb{C}$

In this section we show that the asymptotic fields can be treated as subfields of $^*\mathbb{C}$.

15.1 Definition (Embedding in $^*\mathbb{C}$)**.** Let $\mathcal{M}$ be a convex subring of $^*\mathbb{C}$ and let $\widehat{^*\mathbb{C}} = \widehat{\mathcal{M}}$ is the corresponding asymptotic field (Definition 14.1). For every $\mathbb{M} \in \mathcal{M}ax(\mathcal{M})$ (Definition 14.6) we define the mapping $\sigma_{\mathbb{M}} : \widehat{^*\mathbb{C}} \to {}^*\mathbb{C}$ by $\sigma_{\mathbb{M}}(\widehat{z}) = \text{st}_{\mathbb{M}}(z)$ (or, equivalently, by $\sigma_{\mathbb{M}} = (q_{\mathcal{M}}|\mathbb{M})^{-1}$), where $\text{st}_{\mathbb{M}}$ is the $\mathbb{M}$-standard part mapping (Definition 14.9).

15.2 Theorem (Embedding in $^*\mathbb{C}$)**.** *The mapping* $\sigma_{\mathbb{M}}$ *is a **field embedding** of* $\widehat{^*\mathbb{C}}$ *into* $^*\mathbb{C}$ *with range* $\sigma_{\mathbb{M}}[\widehat{^*\mathbb{C}}] = \mathbb{M}$. *We shall often write simply* $\widehat{^*\mathbb{C}} \subseteq {}^*\mathbb{C}$ *(suppressing the dependence of* $\sigma_{\mathbb{M}}$ *on the choice of* $\mathbb{M}$*). We summarize all these in*

(26) $$\mathbb{C} \subseteq \widehat{^*\mathbb{C}} \subseteq {}^*\mathbb{C} \;\; or \;\; \mathbb{C} \subseteq \widehat{\mathcal{M}} \subseteq {}^*\mathbb{C},$$

depending on the choice of the notation $\widehat{^*\mathbb{C}}$ *or* $\widehat{\mathcal{M}}$ *(part (ii) of Notation 14.2).*

Proof. The fields $\widehat{^*\mathbb{C}}$ and $\mathbb{M}$ are isomorphic by Theorem 14.11 and $\mathbb{M}$ is a subfield of $^*\mathbb{C}$. $\square$

15.3 Remark. According to the above theorem, every maximal field $\mathbb{M}$ determines a unique field embedding $\sigma_{\mathbb{M}}$. Conversely, every field embedding $\sigma_{\mathbb{M}}$ of $\widehat{{}^*\mathbb{C}}$ into ${}^*\mathbb{C}$ determines a maximal field $\mathbb{M} \subset \mathcal{M}$ by $\sigma_{\mathbb{M}}[\widehat{{}^*\mathbb{C}}] = \mathbb{M}$. On the ground of the isomorphism between $\mathbb{M}$ and $\widehat{{}^*\mathbb{C}}$ we shall sometimes identify $\mathbb{M}$ with $\widehat{{}^*\mathbb{C}}$ by simply letting $\mathbb{M} = \widehat{{}^*\mathbb{C}}$. In this environment $\mathrm{st}_{\mathbb{M}}$ reduces to the quotient mapping $q_{\mathcal{M}} : \mathcal{M} \to \widehat{{}^*\mathbb{C}}$. We should note that the embedding of $\widehat{\mathcal{M}}$ into ${}^*\mathbb{C}$ is neither unique, nor canonical because the existence of a maximal field $\mathbb{M}$ depends on the axiom of choice (Lemma 14.7).

Every convex subring $\mathcal{M}$ of ${}^*\mathbb{C}$ determines an asymptotic field $\widehat{\mathcal{M}}$ which we denoted by short by $\widehat{{}^*\mathbb{C}}$ (Notation 14.2). In what follows we shall consider several subrings $\mathcal{M}_1, \mathcal{M}_2$, etc. simultaneously and we shall prefer the more precise notations $\widehat{\mathcal{M}_1}, \widehat{\mathcal{M}_2}$, etc. instead of $\widehat{{}^*\mathbb{C}}$. Now, suppose that $\mathcal{M}_1 \subset \mathcal{M}_2$. Our next goal is to show that there exists a (non-unique, non-canonical) field embedding of $\widehat{\mathcal{M}_1}$ into $\widehat{\mathcal{M}_2}$. We should note that the obvious candidate for a canonical embedding $q_1(z) \to q_2(z)$ of $\widehat{\mathcal{M}_1}$ into $\widehat{\mathcal{M}_2}$ is not defined correctly due to the reverse inclusion of the ideals $\mathcal{M}_{2,0} \subset \mathcal{M}_{1,0}$ (q_1 and q_2 stand for the corresponding quotient mappings).

15.4 Lemma (Synchronized Embeddings)**.** *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two convex subrings of ${}^*\mathbb{C}$ such that $\mathcal{M}_1 \subset \mathcal{M}_2$. Let $\mathcal{M}_{1,0}$ and $\mathcal{M}_{2,0}$ be their maximal ideals and let $\widehat{\mathcal{M}_1} = \mathcal{M}_1/\mathcal{M}_{1,0}$ and $\widehat{\mathcal{M}_2} = \mathcal{M}_2/\mathcal{M}_{2,0}$ be the asymptotic fields generated by $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively (Definition 14.1). Then for every field embedding σ_1 of $\widehat{\mathcal{M}_1}$ into ${}^*\mathbb{C}$ there exists a field embedding σ_2 of $\widehat{\mathcal{M}_2}$ into ${}^*\mathbb{C}$ such that $\sigma_1[\widehat{\mathcal{M}_1}] \subset \sigma_2[\widehat{\mathcal{M}_2}]$. We say that the **embeddings** σ_1 **and** σ_2 **are synchronized**. A similar result holds for every finite many convex subrings $\mathcal{M}_1 \subset \cdots \subset \mathcal{M}_n$ of ${}^*\mathbb{C}$.*

Proof. Let $\mathbb{M}_1 \in Max(\mathcal{M}_1)$ (chosen arbitrarily). There exists $\mathbb{M}_2 \in Max(\mathcal{M}_2)$ such that $\mathbb{M}_1 \subset \mathbb{M}_2$ by Lemma 14.7, since $\mathbb{M}_1 \subseteq \mathcal{M}_2$ by assumption. We let $\sigma_1 = \sigma_{\mathbb{M}_1}$ and $\sigma_2 = \sigma_{\mathbb{M}_2}$ (Definition 15.1). We have $\sigma_1[\widehat{\mathcal{M}_1}] = \mathbb{M}_1 \subset \mathbb{M}_2 = \sigma_2[\widehat{\mathcal{M}_2}]$ by Theorem 15.2. $\qquad\square$

15.5 Theorem (Two Fields)**.** *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two convex subrings of ${}^*\mathbb{C}$ such that $\mathcal{M}_1 \subseteq \mathcal{M}_2$. Then there exists (non-unique) a field embedding $\sigma : \widehat{\mathcal{M}_1} \to \widehat{\mathcal{M}_2}$ of $\widehat{\mathcal{M}_1}$ into $\widehat{\mathcal{M}_2}$. We shall write all this simply as inclusions $\widehat{\mathcal{M}_1} \subset \widehat{\mathcal{M}_2}$. A similar result holds for every finite many convex subrings $\mathcal{M}_1 \subset \cdots \subset \mathcal{M}_n$ of ${}^*\mathbb{C}$.*

Proof. Let σ_1 and σ_2 be two synchronized embeddings of $\widehat{\mathcal{M}_1}$ and $\widehat{\mathcal{M}_2}$ into ${}^*\mathbb{C}$, respectively, in the sense of the above lemma. Then $\sigma : \widehat{\mathcal{M}_1} \to \widehat{\mathcal{M}_2}$, defined by the formula $\sigma = \sigma_2^{-1} \circ \sigma_1$, is a field imbedding of $\widehat{\mathcal{M}_1}$ into $\widehat{\mathcal{M}_2}$. $\square$

15.6 Remark (Non-Canonnical)**.** We should note that the field embedding σ is neither unique, nor canonical in the sense that it can not be determined uniquely in the terms of used in the definitions of $\widehat{\mathcal{M}_1}$ and $\widehat{\mathcal{M}_2}$. Indeed, the embedding σ depends on the choices of maximal fields the existence of which was proved with the help of the Zorn lemma (Lemma 14.7).

16 Examples of Asymptotic Fields

In this section we present several examples of asymptotic fields closely related with the convex rings in Section 12.

16.1 Examples (Examples of Asymptotic Fields)**.** Let ρ be a positive infinitesimal in ${}^*\mathbb{R}$ and let $\mathcal{L}_\rho({}^*\mathbb{C})$, $\mathcal{F}_\rho({}^*\mathbb{C})$, $\mathcal{M}_\rho({}^*\mathbb{C})$ and $\mathcal{E}_\rho({}^*\mathbb{C})$ be the convex subrings of ${}^*\mathbb{C}$ defined in Section 12. We denote by

$$\widehat{\mathcal{F}} = \mathcal{F}/\mathcal{I} = \mathbb{C},$$
$$\widehat{\mathcal{L}_\rho} = \mathcal{L}_\rho/\mathcal{L}_{\rho,0},$$
$$\widehat{\mathcal{F}_\rho} = \mathcal{F}_\rho/\mathcal{I}_\rho,$$
$$\widehat{\mathcal{M}_\rho} = \mathcal{M}_\rho/\mathcal{N}_\rho = {}^\rho\mathbb{C},$$
$$\widehat{\mathcal{E}_\rho} = \mathcal{E}_\rho/\mathcal{E}_{\rho,0},$$
$$\widehat{{}^*\mathbb{C}} = {}^*\mathbb{C}/\{0\} = {}^*\mathbb{C},$$

the corresponding asymptotic fields. We denote by $\mathbb{R}$, $\Re e(\widehat{\mathcal{L}_\rho})$, $\Re e(\widehat{\mathcal{F}_\rho})$, $\Re e({}^\rho\mathbb{C}) = {}^\rho\mathbb{R}$, $\Re e(\widehat{\mathcal{E}_\rho})$ and ${}^*\mathbb{R}$ their real parts, respectively (Definition 14.1). We call the field $\widehat{\mathcal{L}_\rho}$ **multiple logarithmic field** because $\widehat{\log_\nu (1/\rho)}$ is a typical element of $\widehat{\mathcal{L}_\rho}$, where ν is an infinitely large number in ${}^*\mathbb{N}$ (Example 12.4). We call the field $\widehat{\mathcal{F}_\rho}$ **logarithmic field** because $\widehat{\ln \rho}$ is a typical element of $\widehat{\mathcal{F}_\rho}$ (Example 12.5). We call ${}^\rho\mathbb{C}$ **Robinson field of (complex) asymptotic numbers** because $\widehat{\rho^m}$, where $m \in \mathbb{Z}$, and more generally the series of the form $\sum_{n=m}^\infty c_n \widehat{\rho}^n$, where $c_n \in \mathbb{C}$, are typical elements of ${}^\rho\mathbb{C}$ (see the remark below). We call the field $\widehat{\mathcal{E}_\rho}$ **logarithmic-exponential** field because $\widehat{e^{1/\rho}}$ and $\widehat{\ln \rho}$ are typical element of $\widehat{\mathcal{E}_\rho}$ (Example 12.7).

16.2 Theorem. (i) *Each of* $\mathbb{C}, \widehat{\mathcal{L}_\rho}, \widehat{\mathcal{F}_\rho}, \widehat{\mathcal{M}_\rho} = {}^\rho\mathbb{C}, \widehat{\mathcal{E}_\rho}$ *and* ${}^*\mathbb{C}$ *is an* **algebraically closed subfield** *of* ${}^*\mathbb{C}$.

(ii) *There exists an embeddings of each of these fields into* ${}^*\mathbb{C}$ *such that*

$$\mathbb{C} \subset \widehat{\mathcal{L}_\rho} \subset \widehat{\mathcal{F}_\rho} \subset \widehat{\mathcal{M}_\rho} \subset \widehat{\mathcal{E}_\rho} \subset {}^*\mathbb{C}.$$

(iii) *Consequently, each of* $\Re e(\widehat{\mathcal{L}_\rho}), \Re e(\widehat{\mathcal{F}_\rho}), \Re e(\widehat{\mathcal{M}_\rho}) = {}^\rho\mathbb{R}$ *and* $\Re e(\widehat{\mathcal{E}_\rho})$ *is a* **real closed subfield** *of* ${}^*\mathbb{R}$ *and*

$$\mathbb{R} \subset \Re e(\widehat{\mathcal{L}_\rho}) \subset \Re e(\widehat{\mathcal{F}_\rho}) \subset \Re e(\widehat{\mathcal{M}_\rho}) \subset \Re e(\widehat{\mathcal{E}_\rho}) \subset {}^*\mathbb{R}.$$

Proof. (i) The result follows directly from Theorem 14.11 because all these fields are asymptotic fields, i.e. fields of the form $\widehat{\mathcal{M}}$ for some convex subring $\mathcal{M}$ of ${}^*\mathbb{C}$ (Section 12).

(ii) follows from Theorem 15.2, since $\mathcal{F}({}^*\mathbb{C}) = \mathcal{F} \subset \mathcal{L}_\rho \subset \mathcal{F}_\rho \subset \mathcal{M}_\rho \subset \mathcal{E}_\rho \subset {}^*\mathbb{C}$ (see the end of Section 12).

(iii) follows directly from (i) and (ii). $\qquad\qquad\square$

16.3 Example (A. Robinson's Asymptotic Numbers). The field of the **real ρ-asymptotic numbers** ${}^\rho\mathbb{R}$ is introduced by A. Robinson [76] and it is intimately connected with the asymptotic expansions of standard functions (Lightstone&Robinson [57]). The fields ${}^\rho\mathbb{R}$ and ${}^\rho\mathbb{C}$ are also known as **Robinson's valuation fields** because it is endowed with a non-archimedean valuation $v : {}^\rho\mathbb{C} \to \mathbb{R} \cup \{\infty\}$ defined by

$$v(\widehat{z}) = \sup\{r \in \mathbb{Q} \mid z/\rho^r \approx 0\}, \quad \widehat{z} \neq 0,$$

and $v(0) = \infty$. We also have the following **valuation formula** (due to A. Robinson): $v(\widehat{z}) = \mathrm{st}\,(\ln|z|/\ln\rho)$ if $z \in \mathcal{M}_\rho \setminus \mathcal{N}_\rho$ and $v(\widehat{z}) = \infty$ if $z \in \mathcal{N}_\rho$. Notice that $v(\widehat{\rho^x}) = \mathrm{st}(x)$ for every finite number x in ${}^*\mathbb{R}$. The **valuation metric** $d_v : {}^\rho\mathbb{C} \times {}^\rho\mathbb{C} \to \mathbb{R}$ is defined by $d_v(\widehat{z}, \widehat{\zeta}) = e^{-v(\widehat{z-\zeta})}$ under the convention that $e^{-\infty} = 0$. We also define the **valuation norm** $|\widehat{z}|_v = e^{-v(\widehat{z})}$. We should note that the valuation topology and the order topology on ${}^\rho\mathbb{C}$ are the same. Also, the series of the form $\sum_{n=m}^{\infty} c_n \widehat{\rho}^n$ (mentioned earlier) are always convergent. For more recent results on ${}^\rho\mathbb{R}$ we refer to (Todorov&Wolf [97]).

16.4 Example (Logarithmic-Exponential Power Series)**.** The real part $\Re e(\widehat{\mathcal{E}_\rho})$ of $\widehat{\mathcal{E}_\rho}$ is a field extension of the field $\mathbb{R}((t))^{LE}$ of the *logarithmic-exponential power series* introduced in L. Van den Dries, A. MacIntyre and D. Marker [20]). We should mention that the field $\mathbb{R}((t))^{LE}$ has several important applications including the solution of a Hardy's asymptotic open problem. For more on the ordered exponential fields we refer to S. Kuhlmann [45], where the reader will find more references on the subject.

17 Cantor Completeness

We show that all asymptotic fields are Cantor complete and some are algebraically saturated.

17.1 Definition (Cantor Completeness)**.** Let $\mathbb{K}$ be an ordered field (totally ordered field) and κ be an infinite cardinal.

(i) We say that $\mathbb{K}$ is **Cantor κ-complete** if every collection of fewer than κ closed intervals in $\mathbb{K}$ with the finite intersection property (f.i.p.) has a non-empty intersection. We say that $\mathbb{K}$ is simply **Cantor complete** (or, $\mathbb{K}$ is a **semi-η_1-set**) if it is Cantor $\aleph_1$-complete, where $\aleph_1$ is the successor of $\aleph_0 = \mathrm{card}(\mathbb{N})$. This means every nested sequence of closed intervals in $\mathbb{K}$ has a non-empty intersection.

(ii) We say that $\mathbb{K}(i)$ is Cantor κ-complete if $\mathbb{K}$ is Cantor κ-complete.

It is easy to show that a Cantor complete ordered field must be sequentially complete. Two counter-examples to the converse are described in the next section. The terminology for semi-η_1-sets is introduced in Dales&Woodin [17] (see pp.7, 35, 50, 98).

17.2 Theorem (Cantor Completeness)**.** *Let $^*\mathbb{C}$ be κ-saturated for some infinite cardinal κ. Then every asymptotic subfield of $^*\mathbb{C}$ is Cantor κ-complete. In particular, the asymptotic fields $\widehat{\mathcal{L}_\rho}, \widehat{\mathcal{F}_\rho}, {}^\rho\mathbb{C}$ and $\widehat{\mathcal{E}_\rho}$ and their real parts $\Re e(\widehat{\mathcal{L}_\rho}), \Re e(\widehat{\mathcal{F}_\rho}), {}^\rho\mathbb{R}$ and $\Re e(\widehat{\mathcal{E}_\rho})$ (Section 16) are all Cantor κ-complete.*

Proof. Every asymptotic subfield of $^*\mathbb{C}$ is of the form $\widehat{\mathcal{M}}$ for some convex subring $\mathcal{M}$ of $^*\mathbb{C}$ (Definition 14.1). Thus $\widehat{\mathcal{M}} = \mathbb{K}(i)$, where $\mathbb{K} = {}^*\widehat{\mathbb{R} \cap \mathcal{M}}$ is the real part of $\widehat{\mathcal{M}}$. Notice that $^*\widehat{\mathbb{R} \cap \mathcal{M}}$ is a totally ordered field as a real

closed field (Theorem 14.11). We have to show that $\widehat{{}^*\mathbb{R} \cap \mathcal{M}}$ is Cantor κ-complete. Suppose that $\{[a_\gamma, b_\gamma]\}_{\gamma \in \Gamma}$ is a family of closed intervals in $\widehat{{}^*\mathbb{R} \cap \mathcal{M}}$ with the finite intersection property and $\mathrm{card}(\Gamma) < \kappa$. We have $a_\gamma = q_{\mathcal{M}}(\alpha_\gamma)$ for some α_γ in ${}^*\mathbb{R} \cap \mathcal{M}$. Define $B_\gamma = \{\beta \in {}^*\mathbb{R} \cap q_{\mathcal{M}}^{-1}(b_\gamma) : \alpha_\gamma \leq \beta\}$ and observe that $B_\gamma \neq \varnothing$ for each $\gamma \in \Gamma$. Indeed, if $a_\gamma < b_\gamma$, then $B_\gamma = {}^*\mathbb{R} \cap q_{\mathcal{M}}^{-1}(b_\gamma)$. If $a_\gamma = b_\gamma$, then $\alpha_\gamma \in B_\gamma$. Thus (by the axiom of choice) there exists a family $\{\beta_\gamma\}_{\gamma \in \Gamma}$ in ${}^*\mathbb{R} \cap \mathcal{M}$ such that $\beta_\gamma \in B_\gamma$ for all $\gamma \in \Gamma$. As a result, $\{[\alpha_\gamma, \beta_\gamma]\}_{\gamma \in \Gamma}$ is a family of closed intervals in ${}^*\mathbb{R}$ with the finite intersection property. It follows that there exists $x \in {}^*\mathbb{R}$ such that $\alpha_\gamma \leq x \leq \beta_\gamma$ for all $\gamma \in \Gamma$ by the κ-saturation of ${}^*\mathbb{R}$. It is clear that $x \in \mathcal{M}$ by the convexity of $\mathcal{M}$; hence, $a_\gamma \leq q_{\mathcal{M}}(x) \leq b_\gamma$ for all $\gamma \in \Gamma$. $\qquad\square$

17.3 Definition (Algebraic Saturation)**.** Let $\mathbb{K}$ be an ordered field (totally ordered field) and κ be an infinite cardinal.

(i) We say that $\mathbb{K}$ is **algebraically κ-saturated** if every collection of fewer than κ open intervals in $\mathbb{K}$ with the finite intersection property has a non-empty intersection. We say that $\mathbb{K}$ is simply **algebraically saturated** if it is algebraically $\aleph_1$-saturated. This means every nested sequence of open intervals in $\mathbb{K}$ has a non-empty intersection.

(ii) We say that $\mathbb{K}(i)$ is κ-saturated if $\mathbb{K}$ is algebraically κ-saturated.

The fields ${}^*\mathbb{R}$ and ${}^*\mathbb{C}$ are always algebraically κ-saturated for some infinite cardinal κ, since the open intervals in ${}^*\mathbb{R}$ are internal sets.

17.4 Theorem (Algebraically Saturated Fields)**.** *Let ${}^*\mathbb{C}$ be κ-saturated for some infinite cardinal κ. Let (δ_n) be a decreasing generating sequence in ${}^*\mathbb{R}$ and $\mathcal{M}$ be the convex subring of ${}^*\mathbb{C}$ generated by (δ_n) (part (i) of Definition 12.1). Then the asymptotic field $\widehat{\mathcal{M}}$ and its real part $\widehat{{}^*\mathbb{R} \cap \mathcal{M}}$ are algebraically κ-saturated. In particular, the asymptotic fields $\widehat{\mathcal{L}_\rho}$ and $\widehat{\mathcal{F}_\rho}$ as well as their real parts $\Re e(\widehat{\mathcal{L}_\rho})$ and $\Re e(\widehat{\mathcal{F}_\rho})$ (Section 16) are algebraically κ-saturated.*

Proof. Suppose that $(a_\gamma, b_\gamma)_{\gamma \in \Gamma}$ is a family of open intervals in $\widehat{{}^*\mathbb{R} \cap \mathcal{M}}$ with the finite intersection property and $\mathrm{card}(\Gamma) < \kappa$. We have $a_\gamma = \widehat{\alpha_\gamma}$ and $b_\gamma = \widehat{\beta_\gamma}$ for some α_γ and β_γ in ${}^*\mathbb{R} \cap \mathcal{M}$ such that $\alpha_\gamma < \beta_\gamma$ and $\alpha_\gamma - \beta_\gamma \notin \mathcal{M}_0$ for all $\gamma \in \Gamma$. Thus the family of open intervals $(\alpha_\gamma, \beta_\gamma)_{\gamma \in \Gamma}$ in ${}^*\mathbb{R} \cap \mathcal{M}$ has also the finite intersection property. Next, we observe that family of

open intervals $(\alpha_\gamma + 1/\delta_n, \beta_\gamma - 1/\delta_n)_{(\gamma,n)\in\Gamma\times\mathbb{N}}$ in $^*\mathbb{R} \cap \mathcal{M}$ has also the finite intersection property, since $1/\delta_n \in \mathcal{M}_0$ for all $n \in \mathbb{N}$. Thus there exists $x \in {}^*\mathbb{R}$ such that $\alpha_\gamma + 1/\delta_n < x < \beta_\gamma - 1/\delta_n$ for all $\gamma \in \Gamma$ and all $n \in \mathbb{N}$ by the κ-saturation of $^*\mathbb{R}$ since $\mathrm{card}(\Gamma \times \mathbb{N}) = \mathrm{card}(\Gamma)$. Also $x \in \mathcal{M}$ by the convexity of $\mathcal{M}$. Next, we observe that $x - \alpha_\gamma, x - \beta_\gamma \notin \mathcal{M}_0$ for all $\gamma \in \Gamma$ because for each (fixed) $\gamma \in \Gamma$ we have $x - \alpha_\gamma > 1/\delta_n$ and $\beta_\gamma - x > 1/\delta_n$ for all $n \in \mathbb{N}$. Thus $a_\gamma = \widehat{\alpha_\gamma} = \widehat{\alpha_\gamma + 1/\delta_n} < \widehat{x} < \widehat{\beta_\gamma - 1/\delta_n} = \widehat{\beta_\gamma} = b_\gamma$ as required. $\qquad\square$

17.5 Remark (Non-Saturated Fields)**.** We should note that not every asymptotic field is algebraically saturated. For example, the fields $\mathbb{R}$ and $\mathbb{C}$ are certainly not saturated. Let (λ_n) be a increasing generating sequence in $^*\mathbb{R}$ and $\mathcal{M}$ be the convex subring of $^*\mathbb{C}$ generated by (λ_n) (part (ii) of Definition 12.1). Then the asymptotic field $\widehat{\mathcal{M}}$ and its real part $\widehat{^*\mathbb{R} \cap \mathcal{M}}$ are not algebraically saturated. In particular, the asymptotic fields $^\rho\mathbb{C}$ and $\widehat{\mathcal{E}_\rho}$ and their real parts $^\rho\mathbb{R}$ and $\Re e(\widehat{\mathcal{E}_\rho})$ (Section 16) are not algebraically saturated. Indeed, we observe that the nested sequence of open intervals $(0, \widehat{1/\lambda_n})$ in $\widehat{\mathcal{M}}$ has an empty intersection. To show that, suppose (on the contrary) that there exists $x \in \mathcal{M}$ such that $0 < \widehat{x} < \widehat{1/\lambda_n}$ for all $n \in \mathbb{N}$. It follows $0 < x < 1/\lambda_n$ for all $n \in \mathbb{N}$ implying $x \in \mathcal{M}_0$. Thus $\widehat{x} = 0$, a contradiction.

17.6 Theorem (Hypothesis 1)**.** *Let $^*\mathbb{C}$ be κ-saturated. Let $\mathbb{K}$ be a subring of $^*\mathbb{C}$ which is closed under the absolute value in the sense that $z \in \mathbb{K}$ implies $|z| \in \mathbb{K}$. Then the following are equivalent:*

(i) $\mathbb{K}$ *is an asymptotic field (Definition 14.1).*

(ii) $\mathbb{K}$ *is an algebraically closed Cantor κ-complete field.*

Proof. (i)$\Rightarrow$(ii) If $\mathbb{K}$ is an asymptotic field, then $\mathbb{K}$ is algebraically closed Cantor κ-complete by Theorem 14.11 and Theorem 17.2.

(i)$\Leftarrow$(ii) We let $\mathcal{M} = \mathrm{cov}(\mathbb{K})$ (Definition 14.12) and observe that $\mathcal{M}$ is a convex subring of $^*\mathbb{C}$ by Lemma 14.13. In view of Theorem 14.11, to show that $\mathbb{K}$ is an asymptotic field, it suffices to show that $\mathbb{K}$ is a maximal field in $\mathcal{M}$.

$\qquad\square$

18   Power Series in $^*\mathbb{C}$

In this section we show that different fields of generalized power series can be embedded as subfields of $^*\mathbb{C}$. We start with some preliminaries.

1. Let $\mathbb{K}$ be a field. We denote (as usual) by $\mathbb{K}[t]$ the ring of polynomials in one variable with coefficients in $\mathbb{K}$. If $\mathbb{K}$ is ordered, we supply $\mathbb{K}[t]$ with the ordering in which a polynomial is positive if the coefficient in front of the least power of t is positive. We denote by $\mathbb{K}(t)$ the field of rational functions in one variable with coefficients in $\mathbb{K}$. If $\mathbb{K}$ is ordered, then $\mathbb{K}(t)$ is an ordered field under the ordering inherited from $\mathbb{K}[t]$. In this ordering every rational function of the form t^n for some $n \in \mathbb{N}$, is between 0 and every positive element of $\mathbb{K}$. The field $\mathbb{K}$ is naturally embedded in $\mathbb{K}(t)$ by mapping $a \to at^0$.

2. We denote by $\mathbb{K}(t^{\mathbb{Z}})$ the field of **Laurent series** with coefficients in $\mathbb{K}$. If $\mathbb{K}$ is ordered, then $\mathbb{K}(t^{\mathbb{Z}})$ is an ordered field under the ordering in which a series is positive if its leading coefficient is positive. The field $\mathbb{K}(t)$ has a natural embedding into $\mathbb{K}(t^{\mathbb{Z}})$ by $f \to L(f)$, where $L(f)$ is the Laurent expansion of f.

3. Let G an ordered abelian group. For any formal power series $f = \sum_{g \in G} a_g t^g$, where each $a_g \in \mathbb{K}$, the **support** of f is defined by $\operatorname{supp}(f) = \{g \in G : a_g \neq 0\}$. Recall that the **Hahn field** $\mathbb{K}(t^G)$ or $\mathbb{K}((G))$ is the set of all such f's whose support $\operatorname{supp}(f)$ is a well-ordered set (Hahn [33]). We supply $\mathbb{K}(t^G)$ with the ordinary polynomial-like addition and multiplication. The field $\mathbb{K}(t^G)$ has a canonical G-valued Krull valuation in which each non-zero power series is mapped to the least exponent in its support (Krull [44]). If $\mathbb{K}$ is ordered, then $\mathbb{K}(t^G)$ has a natural ordering in which a series is positive if the coefficient corresponding to the least element in its support is positive. This ordering is compatible with the canonical valuation, and is the unique ordering on $\mathbb{K}(t^G)$ in which every positive power of t is between 0 and every positive element of $\mathbb{K}$. Notice that every ordered abelian group contains a copy of $\mathbb{Z}$. Thus $\mathbb{K}(t^{\mathbb{Z}}) \subset \mathbb{K}(t^{\mathbb{R}})$. Summarizing, we have

$$\mathbb{K} \subset \mathbb{K}(t) \subset \mathbb{K}(t^{\mathbb{Z}}) \subset \mathbb{K}(t^G).$$

The field $\mathbb{K}(t^{\mathbb{R}})$ ($\mathbb{K}(t^G)$ is an algebraically closed or real-closed valuation field whenever $\mathbb{K}$ is algebraically closed or real-closed, respectively.

4. Let $G = (\mathbb{R}, +, <)$ be the abelian group of $\mathbb{R}$ with the usual addition and order. In addition to the Hahn field $\mathbb{K}(t^{\mathbb{R}})$, we also consider the field $\mathbb{K}\langle t^{\mathbb{R}}\rangle$ of **Levi-Civita's series** with coefficients in $\mathbb{K}$; the field $\mathbb{K}\langle t^{\mathbb{R}}\rangle$ consists of the series of the form $\sum_{n=0}^{\infty} c_n t^{\nu_n}$, where $c_n \in \mathbb{K}$, which are either finite sums, or (c_n) is a sequence in $\mathbb{K}$, such that $c_n \neq 0$ for all n, and (ν_n) is a strictly increasing unbounded sequence in $\mathbb{R}$. It is clear that $\mathbb{K}\langle t^{\mathbb{R}}\rangle \subset \mathbb{K}(t^{\mathbb{R}})$. Thus we have

$$\mathbb{K} \subset \mathbb{K}(t) \subset \mathbb{K}(t^{\mathbb{Z}}) \subset \mathbb{K}\langle t^{\mathbb{R}}\rangle \subset \mathbb{K}(t^{\mathbb{R}}).$$

The field $\mathbb{K}\langle t^{\mathbb{R}}\rangle$ is an algebraically closed or real-closed valuation field whenever $\mathbb{K}$ is algebraically closed or real-closed, respectively. The field $\mathbb{R}\langle t^{\mathbb{R}}\rangle$ was introduced by Levi-Civita in [54] and later was investigated by Laugwitz in [**?**] as a potential framework for the rigorous foundation of infinitesimal calculus before the advent of Robinson's nonstandard analysis.

5. From Krull [44] and Theorem 2.12 in Luxemburg [58], it is known that every Hahn field of the form $\mathbb{K}(t^{\mathbb{R}})$ is spherically complete in its canonical valuation. In particular, $\mathbb{Q}(t^{\mathbb{R}})$ is spherically complete, hence, sequentially complete. But $\mathbb{Q}(t^{\mathbb{R}})$ is not Cantor complete (for the same reason that $\mathbb{Q}$ is not Cantor complete). Also, $\mathbb{K}\langle t^{\mathbb{R}}\rangle$ is sequentially complete but not spherically complete (Pestov [71], pp. 67).

6. The field $\mathbb{C}\langle t^{\mathbb{R}}\rangle$ is embedded as a subfield of $^{\rho}\mathbb{C}$ by the mapping $\sum_{n=0}^{\infty} c_n t^{\nu_n} \to \sum_{n=0}^{\infty} c_n \rho^{\nu_n}$ (Robinson [76]). The same formula defines a field embedding M_ρ of the Levi-Civita field $\widehat{\mathcal{F}_\rho}\langle t^{\mathbb{R}}\rangle$ into the ring $\mathcal{M}_\rho$ and thus into the field $^{\rho}\mathbb{C}$ (Todorov and Wolf [97], Section 5), where $\widehat{\mathcal{F}_\rho}$ is the logarithmic field defined in Section 16. (We should note that in [97] the real part $\Re e(\widehat{\mathcal{F}_\rho})$ of the field $\widehat{\mathcal{F}_\rho}$ is denoted by $\widehat{^{\rho}\mathbb{R}}$.) We sometimes write simply $t \to \rho$ instead of the more precise M_ρ. Finally, the embedding M_ρ is extended to a field isomorphism between the Hahn field $\widehat{\mathcal{F}_\rho}(t^{\mathbb{R}})$ and $^{\rho}\mathbb{C}$ (Todorov and Wolf [97], Section 6). We shall write this isomorphism simply as equality $\widehat{\mathcal{F}_\rho}(\rho^{\mathbb{R}}) = {}^{\rho}\mathbb{C}$ and summarize all these in

$$\widehat{\mathcal{F}_\rho}(\rho) \subset \widehat{\mathcal{F}_\rho}(\rho^{\mathbb{Z}}) \subset \widehat{\mathcal{F}_\rho}\langle \rho^{\mathbb{R}}\rangle \subset \mathcal{M}_\rho \subset \widehat{\mathcal{F}_\rho}(\rho^{\mathbb{R}}) = {}^{\rho}\mathbb{C}.$$

Also, $\mathbb{C}(\rho^{\mathbb{R}}) \subset \widehat{\mathcal{F}_\rho}(\rho^{\mathbb{R}})$ (trivially) since $\mathbb{C} \subset \widehat{\mathcal{F}_\rho}$. Thus we have also

$$\mathbb{C} \subset \mathbb{C}(\rho) \subset \mathbb{C}(\rho^{\mathbb{Z}}) \subset \mathbb{C}\langle \rho^{\mathbb{R}}\rangle \subset \mathbb{C}(\rho^{\mathbb{R}}) \subset \mathcal{M}_\rho \subset {}^{\rho}\mathbb{C}.$$

18.1 Theorem (Power Series in $^*\mathbb{C}$). *There exists a field embedding of the Hahn field $\widehat{\mathcal{F}}_\rho(\rho^\mathbb{R})$ into $^*\mathbb{C}$. Consequently, we have the embeddings*

$$\widehat{\mathcal{F}}_\rho(\rho) \subset \widehat{\mathcal{F}}_\rho(\rho^\mathbb{Z}) \subset \widehat{\mathcal{F}}_\rho\left\langle\rho^\mathbb{R}\right\rangle \subset \widehat{\mathcal{F}}_\rho\left\langle\rho^\mathbb{R}\right\rangle \subset \mathcal{M}_\rho \subset \widehat{\mathcal{F}}_\rho(\rho^\mathbb{R}) \subset {}^*\mathbb{C},$$
$$\Re e(\widehat{\mathcal{F}}_\rho)(\rho) \subset \Re e(\widehat{\mathcal{F}}_\rho)(\rho^\mathbb{Z}) \subset \Re e(\widehat{\mathcal{F}}_\rho)\left\langle\rho^\mathbb{R}\right\rangle \subset \Re e(\mathcal{M}_\rho) \subset \Re e(\widehat{\mathcal{F}}_\rho)(\rho^\mathbb{R}) \subset {}^*\mathbb{R},$$
$$\mathbb{C}(\rho) \subset \mathbb{C}(\rho^\mathbb{Z}) \subset \mathbb{C}\left\langle\rho^\mathbb{R}\right\rangle \subset \mathbb{C}(\rho^\mathbb{R}) \subset \mathcal{M}_\rho \subset {}^*\mathbb{C},$$
$$\mathbb{R}(\rho) \subset \mathbb{R}(\rho^\mathbb{Z}) \subset \mathbb{R}\left\langle\rho^\mathbb{R}\right\rangle \subset \mathbb{R}(\rho^\mathbb{R}) \subset \Re e(\mathcal{M}_\rho) \subset {}^*\mathbb{R}.$$

Proof. There exists a field embedding of $^\rho\mathbb{C}$ into $^*\mathbb{C}$ by Theorem 16.2. Thus there exists a field embedding of $\widehat{\mathcal{F}}_\rho(\rho^\mathbb{R})$ into $^*\mathbb{C}$ because $\widehat{\mathcal{F}}_\rho(\rho^\mathbb{R})$ and $^\rho\mathbb{C}$ are field isomorphic (Todorov&Wolf [97], Section 6). Also, $\mathbb{C}(\rho^\mathbb{R}) \subset \widehat{\mathcal{F}}_\rho(\rho^\mathbb{R})$ (trivially) since $\mathbb{C} \subset \widehat{\mathcal{F}}_\rho$. $\qquad\square$

We should note that the embedding of $\widehat{\mathcal{F}}_\rho(\rho^\mathbb{R})$ into $^*\mathbb{C}$ is neither canonical, nor unique because the embedding of $^\rho\mathbb{C}$ into $^*\mathbb{C}$ is neither canonical, nor unique (Remark 15.6).

19 Asymptotic Vectors

Let $\mathcal{M}$ be a convex subring of $^*\mathbb{C}$ (Definition 11.1). Let $\widehat{\mathcal{M}}$ be the associated asymptotic field and $\Re e(\widehat{\mathcal{M}})$ be its real part (Definition 14.1). Recall that $\Re e(\widehat{\mathcal{M}})$ can be denoted equivalently by $\widehat{^*\mathbb{R}}$ (see (22) in Notation 14.2). We also let $\widehat{^*\mathbb{R}}^d = \widehat{^*\mathbb{R}} \times \widehat{^*\mathbb{R}} \times \cdots \times \widehat{^*\mathbb{R}}$ (d times).

In this section we present $\widehat{^*\mathbb{R}}^d$ as a factor space in $^*\mathbb{R}^d$. We also discuss the concept of monad in $\widehat{^*\mathbb{R}}^d$.

19.1 Definition (Asymptotic Vectors). Let $\mathcal{M}$ be a convex subring of $^*\mathbb{C}$.

1. We define the linear spaces

$$\mathcal{M}^d(^*\mathbb{R}^d) = \left\{\mathbf{x} \in {}^*\mathbb{R}^d : ||\mathbf{x}|| \in \mathcal{M}\right\},$$
$$\mathcal{M}_0^d(^*\mathbb{R}^d) = \left\{\mathbf{x} \in {}^*\mathbb{R}^d : ||\mathbf{x}|| \in \mathcal{M}_0\right\},$$

where $|| \cdot ||$ stands for the usual Euclidean norm in $^*\mathbb{R}^d$. We define the factor vector space $\widehat{\mathcal{M}^d}(^*\mathbb{R}^d) = \mathcal{M}^d(^*\mathbb{R}^d)/\mathcal{M}_0^d(^*\mathbb{R}^d)$ and denote by $q_\mathcal{M}^d : \mathcal{M}^d(^*\mathbb{R}^d) \to \widehat{\mathcal{M}^d}(^*\mathbb{R}^d)$ the quotient mapping. We shall often write $\widehat{\mathbf{x}}$ instead of $q_\mathcal{M}^d(\mathbf{x})$.

2. We supplied $\widehat{\mathcal{M}^d(^*\mathbb{R}^d)}$ by the addition inherited from $\mathcal{M}^d(^*\mathbb{R}^d)$. We define multiplication between an asymptotic number $\widehat{\lambda} \in \widehat{^*\mathbb{R}}$ and an asymptotic vector $\widehat{\mathbf{x}} \in \widehat{\mathcal{M}^d(^*\mathbb{R}^d)}$ by the formula $\widehat{\lambda}\,\widehat{\mathbf{x}} = \widehat{\lambda\mathbf{x}}$.

3. Let $S \subseteq {}^*\mathbb{R}^d$. We shall sometimes write simply $\widehat{S}$ instead of the more precise $q_{\mathcal{M}}^d[S \cap \mathcal{M}^d(^*\mathbb{R}^d)]$ (suppressing the dependence on $\mathcal{M}$). In this simplified notation we have $\widehat{\mathcal{M}^d(^*\mathbb{R}^d)} = \widehat{^*\mathbb{R}^d}$ (cf. Notation 14.2).

19.2 Lemma (Finite Points). *Let $\mathcal{M}$ be a convex subring of ${}^*\mathbb{C}$ and let $\mathcal{F}(^*\mathbb{R}^d)$ denote the set of the finite points in ${}^*\mathbb{R}^d$. Then $\mathcal{F}(^*\mathbb{R}^d) \subseteq \mathcal{M}^d(^*\mathbb{R}^d)$. Consequently, $\mathbb{R}^d \subseteq \mathcal{M}^d(^*\mathbb{R}^d)$.*

Proof. The result follows immediately from part (i) of Lemma 11.2. □

To the end of this section we shall use $\widehat{^*\mathbb{R}}$, $\widehat{^*\mathbb{R}}^d$, $\widehat{^*\mathbb{R}^d}$ and $\widehat{S}$ instead of the more precise $\Re e(\widehat{\mathcal{M}})$, $(\Re e(\widehat{\mathcal{M}}))^d$, $\widehat{\mathcal{M}^d(^*\mathbb{R}^d)}$ and $q_{\mathcal{M}}^d[S \cap \mathcal{M}^d(^*\mathbb{R}^d)]$, respectively, suppressing the dependence on $\mathcal{M}$ (cf. Notation 14.2).

19.3 Theorem. *Let $\mathcal{M}$ be a convex subring of ${}^*\mathbb{C}$. Then:*

(i) $\widehat{^*\mathbb{R}^d}$ *is a vector space over the field $\widehat{^*\mathbb{R}}$ (22). Also $\widehat{^*\mathbb{R}^d}$ and $\widehat{^*\mathbb{R}}^d$ are isomorphic vector spaces under the mapping $\widehat{\mathbf{x}} \to (\widehat{x_1}, \widehat{x_2}, \ldots, \widehat{x_d})$, where $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathcal{M}^d(^*\mathbb{R}^d)$.*

(ii) $\mathbb{R}^d$ *is a linear subspace of $\widehat{^*\mathbb{R}^d}$ over the field $\mathbb{R}$.*

Proof. The verification is straightforward and we leave it to the reader. □

19.4 Definition (Monads in $\widehat{^*\mathbb{R}^d}$). Let $\Omega \subseteq \mathbb{R}^d$ and let $\mu(\Omega) \subset {}^*\mathbb{R}^d$ be the monad of Ω in ${}^*\mathbb{R}^d$, i.e.

$$(27) \qquad \mu(\Omega) = \{\mathbf{r} + \mathbf{h} : \mathbf{r} \in \Omega,\ \mathbf{h} \in {}^*\mathbb{R}^d, ||\mathbf{h}|| \approx 0\}.$$

Let $\mathcal{M}$ be a convex subring of ${}^*\mathbb{C}$. We define the **monad of Ω in** $\widehat{^*\mathbb{R}^d}$ by the formula $\widehat{\mu(\Omega)} = q_{\mathcal{M}}^d[\mu(\Omega)]$.

Note that $\mu(\Omega) \subseteq \mathcal{M}(^*\mathbb{R}^d)$ by Lemma 19.2, since $\mu(\Omega) \subseteq \mathcal{F}(^*\mathbb{R}^d)$, which guarantees the correctness of the above definition.

19.5 Lemma. *Let $\Omega \subseteq \mathbb{R}^d$ and $\mathcal{M}$ be a convex subring of ${}^*\mathbb{C}$. Then:*

$$(28) \qquad \widehat{\mu(\Omega)} = \{\mathbf{r} + \widehat{\mathbf{h}} : \mathbf{r} \in \Omega,\ \widehat{\mathbf{h}} \in \widehat{^*\mathbb{R}^d}, ||\widehat{\mathbf{h}}|| \approx 0\}.$$

Proof. The verification is straightforward and we leave it to the reader. □

We shall use $\widehat{\mu(\Omega)}$ mostly in the case when Ω is an open subset of $\mathbb{R}^d$.

20 $\mathcal{M}$-Asymptotic Functions

In this section we describe a variety of differential rings $\widehat{\mathcal{M}}(\Omega)$ of generalized functions on an open set Ω of $\mathbb{R}^d$ in terms of a given convex subring $\mathcal{M}$ of $^*\mathbb{C}$ (Section **??**). The elements of $\widehat{\mathcal{M}}(\Omega)$ are named $\mathcal{M}$-**asymptotic functions** because their values are in the field $\widehat{\mathcal{M}}$ of the $\mathcal{M}$-asymptotic numbers and because, more importantly, each $\widehat{\mathcal{M}}(\Omega)$ is an algebra over the field $\widehat{\mathcal{M}}$ (Section **??**). We intend to convert some of $\widehat{\mathcal{M}}(\Omega)$ into **algebras of Colombeau's type** by supplying $\widehat{\mathcal{M}}(\Omega)$ with a copy of the space of Schwartz distributions $\mathcal{D}'(\Omega)$ in one of the next sections. In this section we generalize some of the results in (Oberguggenberger and T. Todorov [68]), where the algebra of ρ-asymptotic functions $^\rho\mathcal{E}(\Omega)$ is introduced; within our more general theory the algebra $^\rho\mathcal{E}(\Omega)$ appears as a particular example (Example 20.5). Similar to some of our results appear in the H. Vernaeve Ph.D. Thesis [100] (for comparison see the definition of $\mathcal{E}_M(\Omega)$ on p. 90, Sec. 3.6).

Here is the **summary** of the basic definitions. The justification of the definitions will be presented later in this section and some of the results will be worked out in detail in some of the next sections.

In what follows $^*\mathbb{C}$ stands for a non-standard extension of the field of the complex numbers $\mathbb{C}$. Let $\mathcal{M}$ be a convex subring in $^*\mathbb{C}$, $\mathcal{M}_0$ be the ideal of the non-invertible elements of $\mathcal{M}$. Let $\widehat{\mathcal{M}}$ be the field of $\mathcal{M}$-asymptotic numbers (Section **??**). Let Ω be an open set of $\mathbb{R}^d$ and let $\mu(\Omega)$ be the monad of Ω and $\mu_{\mathcal{M}}(\Omega)$ denote the $\mathcal{M}$-monad of Ω (30).

20.1 Definition ($\mathcal{M}$-Asymptotic Functions)**.** Then

1. We define the set of $\mathcal{M}$-**moderate functions** $\mathcal{M}(\Omega)$ and the set of the $\mathcal{M}$-**negligible functions** in $^*\mathcal{E}(\Omega)$ by

$$\mathcal{M}(\Omega) = \{f \in {}^*\mathcal{E}(\Omega) \mid (\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega)(\partial^\alpha f(x) \in \mathcal{M})\},$$
$$\mathcal{M}_0(\Omega) = \{f \in {}^*\mathcal{E}(\Omega) \mid (\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega)(\partial^\alpha f(x) \in \mathcal{M}_0)\},$$

   respectively. Let $\widehat{\mathcal{M}}(\Omega) = \mathcal{M}(\Omega)/\mathcal{M}_0(\Omega)$ be the corresponding factor ring. We say that $\widehat{\mathcal{M}}(\Omega)$ **is generated by** $\mathcal{M}$. The elements of $\widehat{\mathcal{M}}(\Omega)$ are named $\mathcal{M}$-**asymptotic functions on** Ω. We denote by $Q_\Omega : \mathcal{M}(\Omega) \to \widehat{\mathcal{M}}(\Omega)$ the corresponding quotient mapping. However we shall often $\widehat{f}$ instead of $Q_\Omega(f)$ for the equivalence class of $f \in \mathcal{M}(\Omega)$.

2. If $S \subseteq {}^*\mathcal{E}(\Omega)$, we shall sometimes suppress the dependence on $\mathcal{M}$ and write simply $\widehat{S}$ instead of the more preices $Q_\Omega[S \cap \mathcal{M}]$. We observe that $\widehat{\mathcal{M}}(\Omega) = \widehat{\mathcal{M}(\Omega)} = \widehat{{}^*\mathcal{E}(\Omega)}$, where in the latter notation, $\widehat{{}^*\mathcal{E}(\Omega)}$, the dependence on $\mathcal{M}$ has been suppressed.

3. We define the **embedding** $\mathcal{E}(\Omega) \hookrightarrow \widehat{\mathcal{M}}(\Omega)$, by the mapping $f \to \widehat{{}^*f}$, where ${}^*f$ is the *non-standard extension* of f.

4. We define a **pairing** between $\widehat{\mathcal{M}}(\Omega)$ and space of test-functions $\mathcal{D}(\Omega)$ by the formula

$$
(29) \qquad \langle \widehat{f}, \ \tau \rangle = q_\mathcal{M} \left(\int_{{}^*\Omega} f(x) \, {}^*\tau(x) \, dx \right),
$$

where $\widehat{f} \in \widehat{\mathcal{M}}(\Omega)$ and $\tau \in \mathcal{D}(\Omega)$ and $q_\mathcal{M} : \mathcal{M} \to \widehat{\mathcal{M}}$ is the quotient mapping (Definition 14.1).

5. Let $\widehat{f}, \widehat{g} \in \widehat{\mathcal{M}}(\Omega)$. We say that $\widehat{f}$ and $\widehat{g}$ are **weakly equal**, and write $\widehat{f} \cong \widehat{g}$, if $\langle \widehat{f}, \ \tau \rangle = \langle \widehat{g}, \ \tau \rangle$ for all $\tau \in \mathcal{D}(\mathbb{R}^d)$. We shall call $\cong$ a **weak equality** in $\widehat{\mathcal{M}}(\Omega)$. Similarly, we say that $\widehat{f}$ and $\widehat{g}$ are **weakly infinitely close** (or simply *infinitely close* for short), and write $\widehat{f} \approx \widehat{g}$, if $\langle \widehat{f}, \ \tau \rangle \approx \langle \widehat{g}, \ \tau \rangle$ in ${}^*\mathbb{C}$ for all $\tau \in \mathcal{D}(\mathbb{R}^d)$. We shall call $\approx$ a **weak infinitesimal relation** in $\widehat{\mathcal{M}}(\Omega)$.

6. Let $\widehat{f} \in \widehat{\mathcal{M}}(\Omega)$ and $\widehat{x} \in \mu_\mathcal{M}(\Omega)$ (30). We define the **value of** $\widehat{f}$ **at** $\widehat{x}$ by the formula $\widehat{f}(\widehat{x}) = \widehat{f(x)}$. We shall use the same notation, $\widehat{f}$, for the corresponding graph $\widehat{f} : \mu_\mathcal{M}(\Omega) \to \widehat{\mathcal{M}}$.

7. Let $\Omega, \mathcal{O}$ be two open sets of $\mathbb{R}^d$ such that $\mathcal{O} \subseteq \Omega$. Let $\widehat{f} \in \widehat{\mathcal{M}}(\Omega)$. We define the **restriction** $\widehat{f} \upharpoonright \mathcal{O}$ of $\widehat{f}$ on $\mathcal{O}$ by the formula

$$
\widehat{f} \upharpoonright \mathcal{O} = \widehat{f|{}^*\mathcal{O}},
$$

where ${}^*\mathcal{O}$ is the non-standard extension of $\mathcal{O}$ and $f|{}^*\mathcal{O}$ is the usual (pointwise) restriction of f on ${}^*\mathcal{O}$.

20.2 Theorem (Some Basic Results)**.** *Let $\mathcal{M}$ be (as before) a convex subring of ${}^*\mathbb{C}$. Then:*

(i) $\mathcal{M}(\Omega)$ *is a differential subring of* $^*\mathcal{E}(\Omega)$ *and* $\mathcal{M}_0(\Omega)$ *is a differential ideal in* $\mathcal{M}(\Omega)$. *Consequently,* $\widehat{\mathcal{M}}(\Omega)$ *is a* **differential ring***.*

(ii) $\mathcal{E}(\Omega)$ *is a* **differential subring** *of* $\widehat{\mathcal{M}}(\Omega)$ *under the embedding* $f \to \widehat{^*f}$*. We shall often write this simply as an inclusion*

$$\mathcal{E}(\Omega) \subset \widehat{\mathcal{M}}(\Omega).$$

(iii) *Let* $\widehat{f}, \widehat{g} \in \widehat{\mathcal{M}}(\Omega)$*. Then* $\widehat{f} = \widehat{g} \Rightarrow \widehat{f} \cong \widehat{g} \Rightarrow \widehat{f} \approx \widehat{g}$*.*

(iv) *The embedding* $f \to \widehat{^*f}$ **preserves the pairing** *between* $\mathcal{E}(\Omega)$ *and* $\mathcal{D}(\Omega)$ *in the sense that for every* $f \in \mathcal{E}(\Omega)$ *and every* $\tau \in \mathcal{D}(\Omega)$ *we have*

$$\int_\Omega f(x)\,\tau(x)\,dx = \left\langle \widehat{^*f}, \tau \right\rangle.$$

Consequently, if $f, g \in \mathcal{E}(\Omega)$*, then either of* $\widehat{f} \cong \widehat{g}$ *or* $\widehat{f} \approx \widehat{g}$ *implies* $f = g$*.*

(v) *The embedding* $\widehat{\mathcal{M}}(\Omega) \hookrightarrow \widehat{\mathcal{M}}^{\,\mu_\mathcal{M}(\Omega)}$*, defined by the pointwise values of* $\widehat{f} \in \widehat{\mathcal{M}}(\Omega)$*, preserves the addition, multiplication and partial differentiation in* $\widehat{\mathcal{M}}(\Omega)$*.*

(vi) *For every* **arcwise connected open set** Ω *of* $\mathbb{R}^d$ *we have*

$$\widehat{\mathcal{M}} = \left\{ \widehat{f} \in \widehat{\mathcal{E}}_\mathcal{M}(\Omega) \mid \nabla \widehat{f} = 0 \right\}.$$

In particular,

$$\widehat{\mathcal{M}} = \left\{ \widehat{f} \in \widehat{\mathcal{E}}_\mathcal{M}(\mathbb{R}^d) \mid \nabla \widehat{f} = 0 \right\}.$$

(vii) *Let* $c \in \mathcal{M}$ *and* $f_c \in {}^*\mathcal{E}(\Omega)$ *denote the constant function* $f_c(x) = c$ *for all* $x \in {}^*\Omega$*. Then the mapping* $\widehat{c} \to \widehat{f}_c$ *from* $\widehat{\mathcal{M}}$ *to* $\widehat{\mathcal{M}}(\Omega)$ *is a differentail ring embedding. Consequently,* $\widehat{\mathcal{M}}(\Omega)$ *is a* **differential algebra over the field** $\widehat{\mathcal{M}}$ *under the ring operations in* $\widehat{\mathcal{M}}(\Omega)$*. In particular the multiplication of functions in* $\widehat{\mathcal{M}}(\Omega)$ *by scalars in* $\widehat{\mathcal{M}}$ *is defined by* $\widehat{c}\,\widehat{f} = \widehat{cf}$*. Also* $\mathcal{E}(\Omega)$ *is a* **differential subalgebra of** $\widehat{\mathcal{M}}(\Omega)$ **over the field** $\mathbb{C}$*. We shall often identify* $\widehat{c}$ *with its image* $\widehat{f}_c$ *and write simply* $\widehat{\mathcal{M}} \subset \widehat{\mathcal{M}}(\Omega)$ *similarly to the more conventional* $\mathbb{C} \subset \mathcal{E}(\Omega)$*.*

(viii) *Let $\mathcal{T}_d$ stand for the usual topology on $\mathbb{R}^d$. The collection $\mathcal{S}_\mathcal{M} =:$ $\{\widehat{\mathcal{M}}(\Omega)\}_{\Omega \in \mathcal{T}_d}$ is a **sheaf** on the topological space $(\mathbb{R}^d, \mathcal{T}_d)$ under the restriction $\upharpoonright$. Consequently, every function $\widehat{f} \in \widehat{\mathcal{M}}(\Omega)$ has a **support** $\mathrm{supp}(\widehat{f})$ which is a **closed set of** Ω.*

Proof: The properties (i)-(v) follow easily from the definition of $\widehat{\mathcal{M}}(\Omega)$ and we shall leave to the reader to check the detail. We shall proof (vi) and (vii) in Section 22 and we shall prove (viii) in Section 23.

Here are **several examples** algebras of asymptotic functions.

20.3 Example ($\mathcal{C}^\infty$-Functions). Let $\mathcal{M} = \mathcal{F}(^*\mathbb{C})$. In this case we have $\mathcal{M}_0 = \mathcal{I}(^*\mathbb{C})$ and $\widehat{\mathcal{M}} = \mathbb{C}$ (Example 12.3). For the $\mathcal{M}$-moderate and $\mathcal{M}$-negligible functions we have $\mathcal{M}(\Omega) = \mathcal{F}(^*\mathcal{E}(\Omega))$ and $\mathcal{M}_0(\Omega) = \mathcal{I}(^*\mathcal{E}(\Omega))$, where

$$\mathcal{F}(^*\mathcal{E}(\Omega)) =: \{f \in {}^*\mathcal{E}(\mathbb{R}^d) \mid (\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega)(\partial^\alpha f(x) \in \mathcal{F}(^*\mathbb{C})),$$
$$\mathcal{I}(^*\mathcal{E}(\Omega)) =: \{f \in {}^*\mathcal{E}(\mathbb{R}^d) \mid (\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega)(\partial^\alpha f(x) \in \mathcal{I}(^*\mathbb{C}))\},$$

respectively. The corresponding ring of $\mathcal{M}$-asymptotic functions

$$\widehat{\mathcal{F}}(\Omega) = \mathcal{F}(^*\mathcal{E}(\Omega))/\mathcal{I}(^*\mathcal{E}(\Omega)),$$

is isomorphic to the ring $\mathcal{E}(\Omega) = \mathcal{C}^\infty(\Omega)$ of the usual $\mathcal{C}^\infty$-functions on Ω.

20.4 Example. Let ρ be (as before) a positive infinitesimal in $^*\mathbb{R}$ and let $\mathcal{M} = \mathcal{F}_\rho$ and $\mathcal{M}_0 = \mathcal{I}_\rho$ (Example 12.5). For the $\mathcal{M}$-moderate and $\mathcal{M}$-negligible functions we have $\mathcal{M}(\Omega) = \mathcal{F}_\rho(^*\mathcal{E}(\Omega))$ and $\mathcal{M}_0(\Omega) = \mathcal{I}_\rho(^*\mathcal{E}(\Omega))$, where

$$\mathcal{F}_\rho(^*\mathcal{E}(\Omega)) =: \left\{f \in {}^*\mathcal{E}(\Omega) \mid (\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega)) \left[\partial^\alpha f(x) \in \mathcal{F}_\rho\right]\right\},$$
$$\mathcal{I}_\rho(^*\mathcal{E}(\Omega)) =: \left\{f \in {}^*\mathcal{E}(\Omega) \mid (\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega)) \left[\partial^\alpha f(x) \in \mathcal{I}_\rho\right]\right\},$$

respectively. The corresponding ring of $\mathcal{M}$-asymptotic functions

$$\widehat{\mathcal{F}_\rho}(\Omega) = \mathcal{F}_\rho(^*\mathcal{E}(\Omega))/\mathcal{I}_\rho(^*\mathcal{E}(\Omega)),$$

is an algebra over the logarithmic field $\widehat{\mathcal{F}_\rho}$ (Example 16.1).

20.5 Example (ρ-Asymptotic Functions). Let ρ be a positive infinitesimal in ${}^*\mathbb{R}$ and let $\mathcal{M} = \mathcal{M}_\rho$ and $\mathcal{M}_0 = \mathcal{N}_\rho$ (Example 12.6). For the $\mathcal{M}$-moderate and $\mathcal{M}$-negligible functions we have $\mathcal{M}(\Omega) = \mathcal{M}_\rho({}^*\mathcal{E}(\Omega))$ and $\mathcal{M}_0(\Omega) = \mathcal{N}_\rho({}^*\mathcal{E}(\Omega))$, where

$$\mathcal{M}_\rho({}^*\mathcal{E}(\Omega)) =: \left\{ f \in {}^*\mathcal{E}(\Omega) \mid (\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega)) \left[\partial^\alpha f(x) \in \mathcal{M}_\rho \right] \right\},$$
$$\mathcal{N}_\rho({}^*\mathcal{E}(\Omega)) =: \left\{ f \in {}^*\mathcal{E}(\Omega) \mid (\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega)) \left[\partial^\alpha f(x) \in \mathcal{N}_\rho \right] \right\},$$

respectively. The corresponding ring of $\mathcal{M}$-asymptotic functions

$$\widehat{\mathcal{M}_\rho}(\Omega) = \mathcal{M}_\rho({}^*\mathcal{E}(\Omega))/\mathcal{N}_\rho({}^*\mathcal{E}(\Omega)),$$

denoted also by ${}^\rho\mathcal{E}(\Omega)$, is an algebra over A. Robinson field ${}^\rho\mathbb{C}$ (Example 16.1). The algebra ${}^\rho\mathcal{E}(\Omega)$ is introduced in (M. Oberguggenberger and T. Todorov[68]) under the name ρ-**asymptotic functions**. We shall follow this terminology. The reader will find a more detail about ${}^\rho\mathcal{E}(\Omega)$ in Chapter **??**. The algebra ${}^\rho\mathcal{E}(\Omega)$ is, in a sense, a non-standard counterpart of a **special Colombeau's algebra** (J. F. Colombeau [10]) with the important **improvement of the properties of the scalars:** The ring of the scalars ${}^\rho\mathbb{C}$ of ${}^\rho\mathcal{E}(\Omega)$ constitutes an algebraically closed Cantor-complete field (Theorem 17.2). In contrast, the ring of the scalars $\widetilde{\mathbb{C}}$ of Colombeau algebra $\mathcal{G}(\Omega)$ is a ring with zero divisors.

20.6 Example (Exponential Asymptotic Functions). Let ρ be (as before) a positive infinitesimal in ${}^*\mathbb{R}$ and let $\mathcal{M} = \mathcal{E}_\rho$ and $\mathcal{M}_0 = \mathcal{E}_{\rho,0}$ (Example 12.7). The corresponding ring of asymptotic functions $\widehat{\mathcal{E}}_\rho(\Omega)$ is an algebra over the exponential field $\widehat{\mathcal{E}}_\rho$ (Example 16.1).

20.7 Example (The case $\mathcal{M} = {}^*\mathbb{C}$). Let $\mathcal{M} = {}^*\mathbb{C}$. In this case $\mathcal{M}_0 = \{0\}$ and $\widehat{\mathcal{M}} = {}^*\mathbb{C}$ (Example 20.7). For the $\mathcal{M}$-moderate and $\mathcal{M}$-negligible functions we have $\mathcal{M}(\Omega) = {}^*\mathcal{E}(\Omega)$ and $\mathcal{M}_0(\Omega) = {}^*\mathcal{E}_0(\Omega)$, respectively, where

$${}^*\mathcal{E}_0(\Omega) = \{ f \in {}^*\mathcal{E}(\mathbb{R}^d) : (\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega)(\partial^\alpha f(x) = 0) \},$$

The ring of the corresponding $\mathcal{M}$-asymptotic functions

$$\widehat{{}^*\mathbb{C}}(\Omega) = {}^*\mathcal{E}(\Omega)/{}^*\mathcal{E}_0(\Omega).$$

is an algebra over the field ${}^*\mathbb{C}$. The algebra $\widehat{{}^*\mathbb{C}}(\Omega)$ is, in a sense, a **non-standard counterpart of Egorov algebra** (Yu. V. Egorov [22]-[23]) with

the important improvement of the properties of the scalars: The ring of the scalars $^*\mathbb{C}$ of $\widehat{^*\mathbb{C}}(\Omega)$ constitutes an algebraically closed saturated field. In contrast, the the scalars of Egorov's algebra are a ring with zero divisors. The algebra $\widehat{^*\mathbb{C}}(\Omega)$ will be studied in detail in Chapter **??**.

21 $\mathcal{M}$-Moderate and $\mathcal{M}$-Negligible Functions

In this section we present **several characterizations** of the $\mathcal{M}$-moderate and $\mathcal{M}$-negligible functions (Section **??**).

Through out this section $\mathcal{M}$ stands for a convex subring of $^*\mathbb{C}$ (Section **??**) and $\mathbb{M} \in \mathcal{M}ax(\mathcal{M})$ stands for a maximal field within $\mathcal{M}$ (Definition 14.6).

21.1 Theorem. *Let $f \in {^*\mathcal{E}}(\Omega)$. Then the following are equivalent:*

(i) $(\forall x \in \mu(\Omega))(f(x) \in \mathcal{M})$.

(ii) $(\forall x \in \mu(\Omega))(\exists M \in \mathbb{M}_+)(|f(x)| \leq M)$.

(iii) $(\forall K \subset\subset \Omega)(\exists M \in \mathbb{M}_+)(\sup_{x \in {^*K}} |f(x)| \leq M)$.

(iv) $(\forall x \in \mu(\Omega))(\exists A \in \mathcal{M} \setminus \mathcal{M}_0)(|f(x)| \leq A)$.

(v) $(\forall K \subset\subset \Omega)(\exists A \in \mathcal{M} \setminus \mathcal{M}_0)(\sup_{x \in {^*K}} |f(x)| \leq A)$.

(vi) $(\forall x \in \mu(\Omega))(\forall B \in {^*\mathbb{R}_+} \setminus \mathcal{M})(|f(x)| < B)$.

(vii) $(\forall K \subset\subset \Omega)(\forall B \in {^*\mathbb{R}_+} \setminus \mathcal{M})(\sup_{x \in {^*K}} |f(x)| < B)$.

21.2 Remark. *We should note that the above theorem remains true even if the maximal field $\mathbb{M}$ is replaced by a set $S \subseteq \mathcal{M} \setminus \mathcal{M}_0$ such that S contains arbitrarily large numbers.*

Proof: (i)$\Leftrightarrow$(ii) follows immediately by part (i) of Theorem 17.6.

(ii)$\Rightarrow$(iii): Let $K \subset\subset \Omega$ and recall that $^*K \subset \mu(\Omega)$ by Theorem 7.3. We observe that $\sup_{\xi \in {^*K}} |f(\xi)| \in \mathcal{M}$. Indeed, suppose (on the contrary) that $\gamma =: \sup_{\xi \in {^*K}} |f(\xi)| \notin \mathcal{M}$ which implies also $\gamma/2 \notin \mathcal{M}$. There exists $y \in {^*K}$ such that $\gamma/2 < |f(y)| < \gamma$ by the choice of γ. It follows $f(y) \notin \mathcal{M}$ which contradicts to (i) (hence it contradicts to (ii)) since $y \in \mu(\Omega)$. On the other hand, $\sup_{\xi \in {^*K}} |f(\xi)| \in \mathcal{M}$ implies that the internal set

$$\mathcal{A} = \{a \in {^*\mathbb{R}_+} : \sup_{\xi \in {^*K}} |f(\xi)| \leq a\},$$

contains $^*\mathbb{R}_+ \setminus \mathcal{M}$ by by part (ii) of Theorem 17.6. Thus $\mathcal{A}$ contains arbitrarily small numbers in $^*\mathbb{C} \setminus \mathcal{M}$. It follows that $\mathcal{A} \cap (\mathcal{M} \setminus \mathcal{M}_0) \neq \varnothing$ by the Underflow of $^*\mathbb{C} \setminus \mathcal{M}$ (Theorem 13.1). Thus $\sup_{x \in {}^*K} |f(x)| \leq A$ holds for any $A \in \mathcal{A} \cap (\mathcal{M} \setminus \mathcal{M}_0)$. Also there exists $M_1 \in \mathbb{M}$ such that $A - M_1 \in \mathcal{M}_0$ by part (i) of Theorem ??. Let $H \in \mathbb{M}_+$. Then (iii) holds for $M = M_1 + H$.

(iii)$\Rightarrow$(iv): Suppose that $x \in \mu(\Omega)$ and observe that $\mathrm{st}(x) \in \Omega$ by the definition of $\mu(\Omega)$. Since Ω is an open set, there exists $\varepsilon \in \mathbb{R}_+$ such that $K \subset\subset \Omega$, where $K = \{r \in \Omega : |r - \mathrm{st}(x)| \leq \varepsilon\}$. There exists $M \in \mathbb{M}_+$ such that $\sup_{\xi \in {}^*K} |f(\xi)| \leq M$ by assumption which implies (iv) for $A = M$ since $x \in {}^*K$ and $M \in \mathbb{M}_+ \subset \mathcal{M} \setminus \mathcal{M}_0$.

The proof of (iv)$\Rightarrow$(v) is almost identical to the proof of (ii)$\Rightarrow$(iii) and we leave it to the reader.

(v)$\Rightarrow$(vi) follows immediately by part (ii) of Theorem 17.6.

(vi)$\Rightarrow$(vii): Suppose (on the contrary) that $\gamma =: \sup_{\xi \in {}^*K} |f(\xi)| \geq B$ for some $K \subset\subset \Omega$ and some $B \in {}^*\mathbb{R}_+ \setminus \mathcal{M}$. We have $B/2 \leq |f(y)| < \gamma$ for some $y \in {}^*K$ by the choice of γ. This contradicts (vi) since $y \in \mu(\Omega)$ and $B/2 \in {}^*\mathbb{R}_+ \setminus \mathcal{M}$.

(vii)$\Rightarrow$(i): Suppose that $x \in \mu(\Omega)$ and observe that $\mathrm{st}(x) \in \Omega$ by the definition of $\mu(\Omega)$. As before there exists $K \subset\subset \Omega$ such that $x \in {}^*K$. As before the internal set $\mathcal{A}$ contains $^*\mathbb{R}_+ \setminus \mathcal{M}$. Thus (as before) $\mathcal{A} \cap (\mathcal{M} \setminus \mathcal{M}_0) \neq \varnothing$ by the Underflow for $^*\mathbb{C} \setminus \mathcal{M}$ (Theorem 13.1). Thus $\sup_{\xi \in {}^*K} |f(\xi)| < A$ for any $A \in \mathcal{A} \cap (\mathcal{M} \setminus \mathcal{M}_0)$. It follows that $|f(x)| < A$ since $x \in {}^*K$ by the choice of K. Thus $f(x) \in \mathcal{M}$ (as required) by the convexity of $\mathcal{M}$.

▲

Here is a **list of characterizations of the $\mathcal{M}$-moderate functions**.

21.3 Corollary ($\mathcal{M}$-Moderate Functions)**.** *Let* $f \in {}^*\mathcal{E}(\Omega)$. *Then the following are equivalent:*

(i) $f \in \mathcal{M}_\mathcal{M}(\Omega)$.

(ii) $(\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega))(\exists M \in \mathbb{M}_+)(|\partial^\alpha f(x)| \leq M)$.

(iii) $(\forall \alpha \in \mathbb{N}_0^d)(\forall K \subset\subset \Omega)(\exists M \in \mathbb{M}_+)(\sup_{x \in {}^*K} |\partial^\alpha f(x)| \leq M)$.

(iv) $(\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega))(\exists A \in \mathcal{M} \setminus \mathcal{M}_0)(|\partial^\alpha f(x)| \leq A)$.

(v) $(\forall \alpha \in \mathbb{N}_0^d)(\forall K \subset\subset \Omega)(\exists A \in \mathcal{M} \setminus \mathcal{M}_0)(\sup_{x \in {}^*K} |\partial^\alpha f(x)| \leq A)$.

(vi) $(\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega))(\forall B \in {}^*\mathbb{R}_+ \setminus \mathcal{M})(|\partial^\alpha f(x)| < B)$.

(vii) $(\forall \alpha \in \mathbb{N}_0^d)(\forall K \subset\subset \Omega)(\forall B \in {}^*\mathbb{R}_+ \setminus \mathcal{M})(\sup_{x \in {}^*K} |\partial^\alpha f(x)| < B)$.

21.4 Remark. *We should note that the above corollary remains true even if the maximal field $\mathbb{M}$ is replaced by a set $S \subseteq \mathcal{M} \setminus \mathcal{M}_0$ such that S contains arbitrarily large numbers.*

Proof: An immediate after replacing f by $\partial^\alpha f$ in Theorem 21.1.

▲

We turn to the $\mathcal{M}$-negligible functions.

21.5 Theorem. *Let $f \in {}^*\mathcal{E}(\Omega)$. Then the following are equivalent:*

(i) $(\forall x \in \mu(\Omega))(f(x) \in \mathcal{M}_0)$.

(ii) $(\forall x \in \mu(\Omega))(\forall M \in \mathbb{M}_+)(|f(x)| < M)$.

(iii) $(\forall K \subset\subset \Omega)(\forall M \in \mathbb{M}_+)(\sup_{x \in {}^*K} |f(x)| < M)$.

(iv) $(\forall x \in \mu(\Omega))(\exists A \in \mathcal{M}_0)(|f(x)| \leq A)$.

(v) $(\forall K \subset\subset \Omega)(\exists A \in \mathcal{M}_0)(\sup_{x \in {}^*K} |f(x)| \leq A)$.

(vi) $(\forall x \in \mu(\Omega))(\forall B \in \mathcal{M} \setminus \mathcal{M}_0)(|f(x)| < |B|)$.

(vii) $(\forall K \subset\subset \Omega)(\forall B \in \mathcal{M} \setminus \mathcal{M}_0)(\sup_{x \in {}^*K} |f(x)| < |B|)$.

21.6 Remark. *We should note that the above theorem remains true even if the maximal field $\mathbb{M}$ is replaced by a set $S \subseteq \mathcal{M} \setminus \mathcal{M}_0$ such that S contains arbitrarily small numbers.*

Proof: We shall prove the equivalence of (i) and (v) only and leave the rest of the proof to the reader (who might decide to adapt the arguments used in the proof of the previous lemma).

(i)$\Rightarrow$(v) Suppose that K is a compact subset of Ω and recall that ${}^*K \subset \mu(\Omega)$ by Theorem 7.3. Notice that $\sup_{x \in {}^*K} |f(x)| \in \mathcal{M}_0$. Indeed, suppose (on the contrary) that $\gamma =: \sup_{x \in {}^*K} |f(x)| \notin \mathcal{M}_0$ which implies $\gamma/2 \notin \mathcal{M}_0$. Also there exists $y \in {}^*K$ such that $\gamma/2 < |f(y)| < \gamma$ by the choice of γ. Thus $|f(y)| \notin \mathcal{M}_0$ contradicting to our assumption (i) since $y \in \mu(\Omega)$. On the other hand, $\sup_{x \in {}^*K} |f(x)| \in \mathcal{M}_0$ implies that the internal set

$$\mathcal{A} = \{c \in {}^*\mathbb{C} : \sup_{x \in {}^*K} |f(x)| \leq |c| \},$$

contains $\mathcal{M} \setminus \mathcal{M}_0$ by by part (ii) of Theorem 17.6. It follows that $\mathcal{A} \cap \mathcal{M}_0 \neq \varnothing$ by the Underflow of $\mathcal{M} \setminus \mathcal{M}_0$ (Theorem 13.1). Thus $\sup_{x \in {}^*K} |f(x)| \leq A$ holds (as required) for any $c \in \mathcal{A} \cap \mathcal{M}_0$ and $A = |c|$.

(i)$\Leftarrow$(v): Suppose that $x \in \mu(\Omega)$. As in the previous lemma, there exists $\varepsilon \in \mathbb{R}_+$ such that $K = \{r \in \Omega : |r - \mathrm{st}(x)| \leq \varepsilon\} \subset\subset \Omega$. Observe that there exists $A \in \mathcal{M}_0$ such that $\sup_{\xi \in {}^*K} |f(\xi)| \leq A$ by assumption. Thus $f(\xi) \in \mathcal{M}_0$ for all $\xi \in {}^*K$ (as required) by the convexity of $\mathcal{M}_0$.

▲

Here is a **list of characterizations of the $\mathcal{M}$-negligible functions**.

21.7 Corollary ($\mathcal{M}$-Negligible Functions). *Let $f \in {}^*\mathcal{E}(\Omega)$. Then the following are equivalent:*

(i) $f \in \mathcal{N}_{\mathcal{M}}(\Omega)$.

(ii) $(\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega))(\forall M \in \mathbb{M}_+)(|f(x)| < M)$.

(iii) $(\forall \alpha \in \mathbb{N}_0^d)(\forall K \subset\subset \Omega)(\forall M \in \mathbb{M}_+)(\sup_{x \in {}^*K} |f(x)| < M)$.

(iv) $(\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega))(\exists A \in \mathcal{M}_0)(|f(x)| \leq A)$.

(v) $(\forall \alpha \in \mathbb{N}_0^d)(\forall K \subset\subset \Omega)(\exists A \in \mathcal{M}_0)(\sup_{x \in {}^*K} |f(x)| \leq A)$.

(vi) $(\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega))(\forall B \in \mathcal{M} \setminus \mathcal{M}_0)(|f(x)| < |B|)$.

(vii) $(\forall \alpha \in \mathbb{N}_0^d)(\forall K \subset\subset \Omega)(\forall B \in \mathcal{M} \setminus \mathcal{M}_0)(\sup_{x \in {}^*K} |f(x)| < |B|)$.

21.8 Remark. *We should note that the above corollary remains true even if the maximal field $\mathbb{M}$ is replaced by a set $S \subseteq \mathcal{M} \setminus \mathcal{M}_0$ such that S contains arbitrarily small numbers.*

Proof: An immediate after replacing f by $\partial^\alpha f$ in Theorem 21.5.

▲

In the next theorem we present several more characterizations of the $\mathcal{M}$-negligible functions (in addition to the presented above), where the quantifier $\forall \alpha \in \mathbb{N}_0^d$ is replaced simply by $\alpha = 0$.

21.9 Theorem (A Simplification). *Let $f \in \mathcal{M}_{\mathcal{M}}(\Omega)$. Then $f \in \mathcal{N}_{\mathcal{M}}(\Omega)$ iff $f(x) \in \mathcal{M}_0$ for all $x \in \mu(\Omega)$. Consequently, we have the following several*

formulas for $\mathcal{N}_\mathcal{M}(\Omega)$:

$\mathcal{N}_\mathcal{M}(\Omega) = \{f \in \mathcal{M}_\mathcal{M}(\Omega) \mid (\forall x \in \mu(\Omega)(f(x) \in \mathcal{M}_0)\}$,
$\mathcal{N}_\mathcal{M}(\Omega) = \{f \in \mathcal{M}_\mathcal{M}(\Omega) \mid (\forall x \in \mu(\Omega))(\forall M \in \mathbb{M}_+)(|f(x)| < M)\}$,
$\mathcal{N}_\mathcal{M}(\Omega) = \{f \in \mathcal{M}_\mathcal{M}(\Omega) \mid (\forall K \subset\subset \Omega)(\forall M \in \mathbb{M}_+)(\sup_{x \in {}^*K} |f(x)| < M)\}$,

$\mathcal{N}_\mathcal{M}(\Omega) = \{f \in \mathcal{M}_\mathcal{M}(\Omega) \mid (\forall x \in \mu(\Omega))(\exists A \in \mathcal{M}_0)(|f(x)| \le A)\}$,
$\mathcal{N}_\mathcal{M}(\Omega) = \{f \in \mathcal{M}_\mathcal{M}(\Omega) \mid (\forall K \subset\subset \Omega)(\exists A \in \mathcal{M}_0)(\sup_{x \in {}^*K} |f(x)| \le A)\}$,

$\mathcal{N}_\mathcal{M}(\Omega) = \{f \in \mathcal{M}_\mathcal{M}(\Omega) \mid (\forall x \in \mu(\Omega))(\forall B \in \mathcal{M} \setminus \mathcal{M}_0)(|f(x)| < |B|)\}$,
$\mathcal{N}_\mathcal{M}(\Omega) = \{f \in \mathcal{M}_\mathcal{M}(\Omega) \mid (\forall K \subset\subset \Omega)(\forall B \in \mathcal{M} \setminus \mathcal{M}_0)(\sup_{x \in {}^*K} |f(x)| < |B|)\}$.

Proof: ($\Rightarrow$) follows immediately after letting $\alpha = 0$.

($\Leftarrow$) Suppose that $x \in \mu(\Omega)$. We have to show that $\partial^\alpha f(x) \in \mathcal{M}_0$ for all multi-indexes $\alpha \in \mathbb{N}_0^d$, $|\alpha| \ge 1$. We start with $|\alpha| = 1$. If $\nabla f(x) = 0$, there is nothing to prove. Suppose that $\nabla f(x) \ne 0$ and let $\varepsilon \in \mathbb{M}_+$. It suffices to show that $||\nabla f(x)|| < \varepsilon$ in view of Theorem 17.6. Since Ω is an open set, there exists an open relatively compact set $\mathcal{O}$ of Ω such that $\text{st}(x) \in \mathcal{O} \subset\subset \Omega$. Now $f \in \mathcal{M}_\mathcal{M}(\Omega)$ implies $\left|\sum_{|\alpha|=2} \partial^\alpha f(\xi)\right| < \delta$ for some $\delta \in \mathbb{M}_+$ and all $\xi \in {}^*\mathcal{O}$ by Corollary 21.3 since ${}^*\mathcal{O} \subset \mu(\Omega)$. Let $h \in \mathcal{I}(\mathbb{M}^d)$ be an infinitesimal vector with the direction of $\nabla f(x)$ and of length $||h|| < \varepsilon/\delta$. Notice that $||h|| \in \mathbb{M}_+$ thus $||h|| \in \mathcal{M} \setminus \mathcal{M}_0$ which is important for what follows. We have $|f(x+h) - f(x)| < \delta ||h||^2/2$ by part (vi) of Theorem 17.6 since $f(x+h) - f(x) \in \mathcal{M}_0$ by assumption and $x + h \in \mu(\Omega)$. Next we observe that the Taylor formula:

$$\nabla f(x) \cdot h = f(x+h) - f(x) - \frac{1}{2} \sum_{|\alpha|=2} \partial^\alpha f(x + \theta h) \, h^\alpha.$$

holds for some $\theta \in {}^*\mathbb{R}$, $0 < \theta < 1$, by Transfer Principle (Theorem 4.8). Thus $x + \theta h \approx x \approx \text{st}(x)$ implying $x + \theta h \in {}^*\mathcal{O}$. We have

$$|\nabla f(x) \cdot h| < \delta ||h||^2/2 + \delta ||h||^2/2 < \delta ||h||^2.$$

Also we have $|\nabla f(x) \cdot h| = ||\nabla f(x)|| \, ||h||$ by the choice of the direction of h. It follows $||\nabla f(x)|| = \delta ||h|| < \varepsilon$ as required. We generalize this result for $|\alpha| = 2, 3, \ldots$ by induction. The different formulas for $\mathcal{N}_\mathcal{M}(\Omega)$ follow immediately by Theorem 21.5. ▲

22 Pointwise Values and Fundamental Theorem

Recall that every non-standard smooth function $f \in {}^*\mathcal{E}(\Omega)$ can be characterized as a pointwise function of the form $f : {}^*\Omega \to {}^*\mathbb{C}$ in the sense that there exists an embedding ${}^*\mathcal{E}(\Omega) \hookrightarrow {}^*\mathbb{C}^{{}^*\Omega}$ which preserves the ring operations and the partial differentiation of any order (Section 8). Among other things the purpose of this section is to show that every asymptotic function $\widehat{f} \in \widehat{\mathcal{E}_\mathcal{M}}(\Omega)$ (Section **??**) can be characterized as a pointwise function of the form $\widehat{f} : \mu_\mathcal{M}(\Omega) \to \widehat{\mathcal{M}}$ in the sense that there exists an embedding $\widehat{\mathcal{E}_\mathcal{M}}(\Omega) \hookrightarrow \widehat{\mathcal{M}}^{\mu_\mathcal{M}(\Omega)}$ which preserves the ring operations and the partial differentiation of any order. We also prove a fundamental theorem of calculus in $\widehat{\mathcal{E}_\mathcal{M}}(\Omega)$.

We shall use the notation introduced in the first several pages in (Section **??**) and (Section **??**). In particular, $\mathcal{M}$ stands for a convex subring of ${}^*\mathbb{C}$ (Section **??**). If $\Omega \subseteq \mathbb{R}^d$ is an open set of $\mathbb{R}^d$, then

$$(30) \qquad \mu_\mathcal{M}(\Omega) = \{ r + dx \mid r \in \Omega,\ dx \in \Re(\widehat{\mathcal{M}}^d),\ ||dx|| \approx 0 \},$$

is the $\mathcal{M}$-monad of Ω. Here $\Re(\widehat{\mathcal{M}}^d)$ stands for the real part of the vector space $\widehat{\mathcal{M}}^d$. We denote by $\widehat{\mathcal{M}}^{\mu_\mathcal{M}(\Omega)}$ the ring of the functions F of the form $F : \mu_\mathcal{M}(\Omega) \to \widehat{\mathcal{M}}$ (Section **??**).

In this section we generalize some of the results in Todor Todorov [95] where the particular case $\mathcal{M} = \mathcal{M}_\rho({}^*\mathbb{C})$ (Example 16.3) is discussed only. The closest counterpart in J.F. Colombeau's theory can be found in M. Kunzinger and M. Oberguggenberger's article [46], where a characterization of Colombeau's generalized functions in $\mathcal{G}(\Omega)$ in the ring of generalized scalars $\widetilde{\mathbb{C}}$ is established.

For convenience of the reader we shall recall the definition pointwise values presented in (Section **??**).

22.1 Definition (Pointwise Values)**.** Let $\widehat{f} \in \widehat{\mathcal{E}_\mathcal{M}}(\Omega)$ be a $\mathcal{M}$-asymptotic function (Section **??**) and $\widehat{x} \in \mu_\mathcal{M}(\Omega)$ be a $\mathcal{M}$-asymptotic point. We define the **value of** $\widehat{f}$ **at** $\widehat{x}$ by the formula

$$\widehat{f}(\widehat{x}) = \widehat{f(x)}.$$

We shall use the same notation, $\widehat{f}$, for the asymptotic function $\widehat{f} \in \widehat{\mathcal{E}_{\mathcal{M}}}(\Omega)$ and its graph $\widehat{f} \in \widehat{\mathcal{M}}^{\mu_{\mathcal{M}}(\Omega)}$ given by the mapping $\widehat{f} : \mu_{\mathcal{M}}(\Omega) \to \widehat{\mathcal{M}}$.

The correctness of the above definition is justified by the following result.

22.2 Lemma (Correctness). *Let $x, y \in \mu(\Omega)$ and $f, g \in \mathcal{M}_{\mathcal{M}}(\Omega)$. Then $x - y \in \mathcal{M}_0$ and $f - g \in \mathcal{N}_{\mathcal{M}}(\Omega)$ implies $f(x) - g(y) \in \mathcal{M}_0$.*

Proof: We have $f(x) - f(y) = \nabla f(t) \cdot (x - y)$ by Transfer Principle (Theorem 4.8) for some $t \in {}^*\mathbb{R}^d$ between x and y (in the sense that $t = x + \theta(y - x)$ for some $\theta \in {}^*\mathbb{R}$, $0 < \theta < 1$). Also

$$|f(x) - g(y)| = |f(x) - f(y) + f(y) - g(y)| \leq |f(x) - f(y)| + |f(y) - g(y)| \leq$$
$$\leq ||\nabla f(t)|| \, ||x - y|| + |f(y) - g(y)|.$$

Observe that $x - y \in \mathcal{M}_0$ implies $x - y \approx 0$ by part (iii) of Theorem 17.6) implying $\mathrm{st}(x) = \mathrm{st}(y) = \mathrm{st}(t)$. It follows $t \in \mu(\Omega)$ since $x, y \in \mu(\Omega)$ by assumption. Thus $f \in \mathcal{M}_{\mathcal{M}}(\Omega)$ implies $||\nabla f(t)|| \in \mathcal{M}$. For the first term we have $||\nabla f(t)|| \, ||x - y|| \in \mathcal{M}_0$ since $||x - y|| \in \mathcal{M}_0$ by assumption and $\mathcal{M}_0$ is an ideal in $\mathcal{M}$. Also $f - g \in \mathcal{N}_{\mathcal{M}}(\Omega)$ implies $|f(y) - g(y)| \in \mathcal{M}_0$ since $y \in \mu(\Omega)$ by assumption. Thus $|f(x) - g(y)| \in \mathcal{M}_0$ as required. ▲

Here is another similar result which plays some role in what follows.

22.3 Lemma. *Let $x \in \mu(\Omega)$ and $f \in \mathcal{M}_{\mathcal{M}}(\Omega)$. Then:*

(i) $h \in \mathcal{M}_0$ *implies* $f(x + h) - f(x) \in \mathcal{M}_0$.

(ii) $h \in \mathcal{M}_0$ *and* $h \neq 0$ *implies* $\frac{|f(x+h) - f(x) - \nabla f(x) \cdot h|}{||h||} \in \mathcal{M}_0$.

Proof: (i) follows directly from the previous lemma for $y = x + h$ and $f = g$.

(ii) By the Mean Value Theorem applied by Transfer Principle (Theorem 4.8), we have $\nabla f(x) \cdot h = f(x + h) - f(x) - \frac{1}{2} \sum_{|\alpha|=2} \partial^\alpha f(x + \theta h) h^\alpha$ for some $\theta \in {}^*\mathbb{R}$, $0 < \theta < 1$. Thus we have

$$\frac{|f(x+h) - f(x) - \nabla f(x) \cdot h|}{||h||} \leq \frac{1}{2} \sum_{|\alpha|=2} |\partial^\alpha f(x + \theta h)| \, ||h|| \in \mathcal{M}_0,$$

as required, because $\mathcal{M}_0$ is an ideal in $\mathcal{M}$ and $\partial^\alpha f(x + \theta h) \in \mathcal{M}$ by assumption since $x + \theta h \in \mu(\Omega)$.
  ▲

Recall that we have the embedding $\mathcal{E}(\Omega) \hookrightarrow \widehat{\mathcal{E}}(\Omega)$ under the mapping $f \to {}^*\widehat{f}$ (Section **??**). The next result shows that the evaluation in $\widehat{\mathcal{E}_{\mathcal{M}}}(\Omega)$ reduces to the usual evaluation in $\mathcal{E}(\Omega)$. Recall that

22.4 Proposition (The Usual Evaluation). *Let $f \in \mathcal{E}(\Omega)$. Then $\widehat{{}^*f}$ is an extension of f, i.e. $\widehat{{}^*f} \,|\, \Omega = f$.*

Proof: $\widehat{{}^*f}(\widehat{x}) = \widehat{{}^*f(x)} = \widehat{f(x)} = f(x)$ since ${}^*f$ is an extension of f. We also have $x = \widehat{x}$ for all $x \in \Omega$ by the identification Ω with its image in $\Re(\mathcal{M}^d)$ (# 22, Section **??**). Thus $\widehat{{}^*f}(x) = f(x)$ as required. ▲

In what follows the cardinal number κ stands for the saturation of ${}^*\mathbb{C}$ (Section 2). Recall that $\kappa = \operatorname{card}(\mathcal{I})$, where $\mathcal{I}$ is the index set used in the construction ${}^*\mathbb{C}$ (Section 4).

22.5 Theorem (Differential Ring Embedding). *The mapping*

$$\widehat{\mathcal{E}_{\mathcal{M}}}(\Omega) \ni \widehat{f} \to \widehat{f} \in \widehat{\mathcal{M}}^{\mu_{\mathcal{M}}(\Omega)},$$

*from $\widehat{\mathcal{E}_{\mathcal{M}}}(\Omega)$ into $\widehat{\mathcal{M}}^{\mu_{\mathcal{M}}(\Omega)}$ is a **differential ring embedding** in the sense that it is injective and preserves the ring operations and partial differentiation of any order.*

22.6 Remark (Interpretation). *Recall that $\mu_{\mathcal{M}}(\Omega) \subset \Re(\widehat{\mathcal{M}}^d)$ and thus $(\mu_{\mathcal{M}}(\Omega), T_<)$ is a topological space. Similarly, $(\widehat{\mathcal{M}}, T_<)$ is a topological space (Section **??**). With this in mind, let $\mathcal{C}^\infty(\mu_{\mathcal{M}}(\Omega), \widehat{\mathcal{M}})$ denote the space of the $\mathcal{C}^\infty$-functions from $\mu_{\mathcal{M}}(\Omega)$ into $\widehat{\mathcal{M}}$. The above theorem shows that $\widehat{\mathcal{E}_{\mathcal{M}}}(\Omega)$ is isomorphic to $\mathcal{C}^\infty(\mu_{\mathcal{M}}(\Omega), \widehat{\mathcal{M}})$. Based on this result we shall sometimes identify a given asymptotic function with its graph and write simply $\widehat{\mathcal{E}_{\mathcal{M}}}(\Omega) = \mathcal{C}^\infty(\mu_{\mathcal{M}}(\Omega), \widehat{\mathcal{M}})$ or $\widehat{\mathcal{E}_{\mathcal{M}}}(\Omega) \subset \widehat{\mathcal{M}}^{\mu_{\mathcal{M}}(\Omega)}$ instead of the more precise $\widehat{\mathcal{E}_{\mathcal{M}}}(\Omega) \hookrightarrow \widehat{\mathcal{M}}^{\mu_{\mathcal{M}}(\Omega)}$. We should note that $\widehat{\mathcal{M}}^{\mu_{\mathcal{M}}(\Omega)} \setminus \widehat{\mathcal{E}_{\mathcal{M}}}(\Omega) \neq \varnothing$.*

Proof: To show that the mapping is injective, observe that $\widehat{f}(\widehat{x}) = 0$ for all $\widehat{x} \in \mu_{\mathcal{M}}(\Omega)$ is equivalent to $f(x) \in \mathcal{M}_0$ for all $\forall x \in \mu(\Omega)$. The latter implies $f \in \mathcal{N}_{\mathcal{M}}(\Omega))$ by Theorem 21.9. Thus $\widehat{f} = 0$ as required. The mapping preserves the addition because $(\widehat{f} + \widehat{g})(\widehat{x}) = \widehat{f}(\widehat{x}) + \widehat{g}(\widehat{x}) = \widehat{f(x) + g(x)}$ and similarly for the multiplication. We turn to the preserving of the partial differentiation. Let $x \in \mu(\Omega)$ and $f \in \mathcal{M}_{\mathcal{M}}(\Omega)$. In view of the fact that every maximal field $\mathbb{M} \in Max(\mathcal{M})$ (Definition 14.6) is isomorphic to $\widehat{\mathcal{M}}$ (Lemma 14.4), it suffices to show that for every $\varepsilon \in \mathbb{M}_+$ there exists $\delta \in \mathbb{M}_+$ such that for every $h \in {}^*\mathbb{R}^d$ we have:

(a) $||h|| < \delta$ implies $|f(x+h) - f(x)| < \varepsilon$.

(b) $0 < ||h|| < \delta$ implies $\frac{|f(x+h)-f(x)-\nabla f(x)\cdot h|}{||h||} < \varepsilon$

We have to consider separately two differently cases: Suppose first, that $\widehat{\mathcal{M}}$ has a base for the open neighborhoods of the zero of cardinality less than κ. Since $\widehat{\mathcal{M}}$ and $\mathbb{M}$ are isomorphic, it follows that there exists a set $\Gamma \subseteq \mathbb{M}_+$ of cardinality less than κ such that the collection of open intervals $(0, \gamma), \gamma \in \Gamma$, is a base for the open neighborhoods of the zero in $\mathbb{M}_+$. Now, suppose (on the contrary) that (a) and (b) fail, i.e. there exists $\varepsilon \in \mathbb{M}_+$ such that for every $\delta \in \Gamma$ we have $X_\delta \neq \varnothing$ and $Y_\delta \neq \varnothing$, where

$$X_\delta = \left\{ h \in {}^*\mathbb{R}^d : ||h|| < \delta \text{ and } |f(x+h) - f(x)| > \varepsilon \right\},$$
$$Y_\delta = \left\{ h \in {}^*\mathbb{R}^d : 0 < ||h|| < \delta \text{ and } \frac{|f(x+h) - f(x) - \nabla f(x) \cdot h|}{||h||} > \varepsilon \right\}.$$

We observe that the families $\{X_\delta\}_{\delta \in \Gamma}$ and $\{Y_\delta\}_{\delta \in \Gamma}$ have the finite intersection properties. Thus there exist $h_1, h_2 \in {}^*\mathbb{R}^d$ such that $h_1 \in X_\delta$ and $h_2 \in Y_\delta$ for all $\delta \in \Gamma$ by the Saturation Principle (Theorem 4.4). It follows that $||h_1||, ||h_1|| \in \mathcal{M}_0$ and $f(x+h_1) - f(x) \notin \mathcal{M}_0$ and $\frac{|f(x+h)-f(x)-\nabla f(x)\cdot h|}{||h||} \notin \mathcal{M}_0$ by Theorem 17.6 contradicting the result of Lemma 22.3. This proves the preservation of the partial derivatives ∂^α for $|\alpha| \leq 1$. The generalization of the result to all multi-indices α follow by induction. Suppose now that $\widehat{\mathcal{M}}$ does not have a base for the open neighborhoods of the zero of cardinality less than κ. In this case we have $\widehat{\mathcal{E}_{\mathcal{M}}}(\Omega) = {}^*\mathcal{E}(\Omega)$ (Example **??**). Thus the preservation of the partial differentiation follows by default since ${}^*\mathcal{E}(\Omega)$ consists exactly of the $\mathcal{C}^\infty$-functions from ${}^*\Omega$ into ${}^*\mathbb{C}$. ▲

22.7 Theorem (Fundamental Theorem). *Let Ω be an arcwise connected open set of $\mathbb{R}^d$ and let $f \in \mathcal{M}_{\mathcal{M}}(\Omega)$. Then the following are equivalent:*

(i) $(\exists \widehat{c} \in \widehat{\mathcal{M}})(\forall \widehat{x} \in \mu_{\mathcal{M}}(\Omega))(\widehat{f}(\widehat{x}) = \widehat{c})$.

(ii) $(\exists c \in \mathcal{M})(\forall x \in \mu(\Omega))(f(x) - c \in \mathcal{M}_0)$.

(iii) $(\forall x \in \mu(\Omega))(||\nabla f(x)|| \in \mathcal{M}_0)$.

(iv) $(\forall \widehat{x} \in \mu_{\mathcal{M}}(\Omega))(\nabla \widehat{f}(\widehat{x}) = 0)$.

(v) $\nabla \widehat{f} = 0$ *in* $\widehat{\mathcal{E}_{\mathcal{M}}}(\Omega)$.

Proof: (i)⇔(ii), (iii)⇔(iv) and (iv)⇔(v) follow directly from Theorem 22.5.

(ii)⇒(iii): Suppose that $x \in \mu(\Omega)$. If $\nabla f(x) = 0$, there is nothing to prove. Suppose that $\nabla f(x) \neq 0$ and let $h \in \mathcal{I}(\mathbb{M}^d)$ be an infinitesimal vector in the direction of $\nabla f(x)$. By the Mean Value Theorem applied by Transfer Principle (Theorem 4.8), we have

$$\nabla f(x) \cdot h = f(x+h) - f(x) - \frac{1}{2} \sum_{|\alpha|=2} \partial^\alpha f(x + \theta h) \, h^\alpha,$$

for some $\theta \in {}^*\mathbb{R}$, $0 < \theta < 1$. We have $\left| \frac{1}{2} \sum_{|\alpha|=2} \partial^\alpha f(x + \theta h) \right| \leq \delta$ for some $\delta \in \mathbb{M}_+$ by Theorem 17.6 since $x + \theta h \in \mu(\Omega)$ and $f \in \mathcal{M}_\mathcal{M}(\Omega)$ by assumption. Also $|\nabla f(x) \cdot h| = ||\nabla f(x)|| \, ||h||$ by the choice of the direction of h. Thus

$$||\nabla f(x)|| \leq \left(\frac{f(x+h) - f(x)}{||h||^2} + \delta \right) ||h||,$$

Observe that $f(x+h) - f(x) \in \mathcal{M}_0$ by assumption since $x + h \in \mu(\Omega)$. Thus $\frac{f(x+h)-f(x)}{||h||^2} + \delta \in \mathbb{M}_+$. Consequently, there exists $M \in \mathbb{M}_+$ such that the internal set

$$\mathcal{A} = \left\{ ||h|| \; : \; h \in {}^*\mathbb{R}^d, \; \frac{\nabla f(x)}{||\nabla f(x)||} = \frac{h}{||h||}, \; ||\nabla f(x)|| \leq M \, ||h|| \right\},$$

contains $\mathcal{I}(\mathbb{M}_+)$. Thus $\mathcal{A}$ contains arbitrarily small numbers in $\mathcal{M} \setminus \mathcal{M}_0$ since $\mathbb{M}_+ \subset \mathcal{M} \setminus \mathcal{M}_0$. It follows that $\mathcal{A}$ contains arbitrarily large numbers $\mathcal{M}_0$ by the Underflow of $\mathcal{M} \setminus \mathcal{M}_0$ (Theorem 13.1). Thus there exists $h \in {}^*\mathbb{R}^d$ such that $||\nabla f(x)|| \leq M \, ||h||$ and $||h|| \in \mathcal{M}_0$. It follows that $||\nabla f(x)|| \in \mathcal{M}_0$ (as required) since $\mathcal{M}_0$ is an ideal in $\mathcal{M}$.

(ii)⇐(iii): Suppose that $x, y \in \mu(\Omega)$. Since Ω is arcwise connected by assumption, it follows that ${}^*\Omega$ is ∗-arcwise connected by Transfer Principle (Theorem 4.8). Thus there exists a ∗-continuous curve $L \subset \mu(\Omega)$ which connects x and y. We have

$$f(x) - f(y) = \int_L \nabla f(t) \cdot dl,$$

(again, by Transfer Principle). It follows that

$$f(x) - f(y) = \nabla f(t) \cdot (x - y),$$

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for some $t \in L$ by the Mean Value Theorem (and Transfer Principle). Thus $|f(x) - f(y)| \leq ||\nabla f(t)|| \, ||x - y|| \in \mathcal{M}_0$, since (as before) $\mathcal{M}_0$ is an ideal in $\mathcal{M}$ and we have $||\nabla f(t)|| \in \mathcal{M}_0$ by assumption and $||x - y|| \in \mathcal{M}(^*\mathbb{R}) \subset \mathcal{M}$. Let $c = f(y)$ for some (any) $y \in \mu(\Omega)$. The result is $f(x) - c \in \mathcal{M}_0$ for all $x \in \mu(\Omega)$ as required. $\blacktriangle$

22.8 Corollary (Constant Functions)**.** *Let Ω be an arcwise connected open set of $\mathbb{R}^d$. Then*

$$(31) \qquad\qquad \widehat{\mathcal{M}} = \left\{ \widehat{f} \in \widehat{\mathcal{E}}_{\mathcal{M}}(\Omega) \mid \nabla \widehat{f} = 0 \right\},$$

In particular,

$$(32) \qquad\qquad \widehat{\mathcal{M}} = \left\{ \widehat{f} \in \widehat{\mathcal{E}}_{\mathcal{M}}(\mathbb{R}^d) \mid \nabla \widehat{f} = 0 \right\}.$$

Proof: The inclusion $\widehat{\mathcal{M}} \subseteq \left\{ \widehat{f} \in \widehat{\mathcal{E}}_{\mathcal{M}}(\Omega) \mid \nabla \widehat{f} = 0 \right\}$ follows directly from Theorem 22.7 in view of the embedding $\widehat{\mathcal{M}} \hookrightarrow \widehat{\mathcal{E}}_{\mathcal{M}}(\Omega)$ (through constant functions) descussed in part (v) of Theorem 20.2. The inclusion $\left\{ \widehat{f} \in \widehat{\mathcal{E}}_{\mathcal{M}}(\Omega) \mid \nabla \widehat{f} = 0 \right\} \subseteq \widehat{\mathcal{M}}$ follows also from Theorem 22.7 and the identification of the constant functions in $\widehat{\mathcal{E}}_{\mathcal{M}}(\Omega)$ with their values. $\blacktriangle$

23 Local Properties of Asymptotic Functions

23.1 Definition (Restriction)**.** Let $\Omega, \mathcal{O}$ be two open sets of $\mathbb{R}^d$ such that $\mathcal{O} \subseteq \Omega$. Let $\widehat{f} \in \widehat{\mathcal{E}}_{\mathcal{M}}(\Omega)$. We define the **restriction** $\widehat{f} \restriction \mathcal{O}$ of $\widehat{f}$ on $\mathcal{O}$ by the formula

$$\widehat{f} \restriction \mathcal{O} = \widehat{f \, |^*\mathcal{O}},$$

where $^*\mathcal{O}$ is the non-standard extension of $\mathcal{O}$ and $f \, |^*\mathcal{O}$ is the usual (pointwise) restriction of f on $^*\mathcal{O}$ (Section 4).

The above definition is justified by the following result.

23.2 Lemma (Justification)**.** *Let $f, g \in \mathcal{M}_{\mathcal{M}}(\Omega)$ and $f - g \in \mathcal{N}_{\mathcal{M}}(\Omega)$. Then $f \, |^*\mathcal{O} - g \, |^*\mathcal{O} \in \mathcal{N}_{\mathcal{M}}(\mathcal{O})$.*

Proof: For every $x \in \mu(\mathcal{O})$ we have $f(x) - g(x) \in \mathcal{M}_0$ by assumption since $\mu(\mathcal{O}) \subseteq \mu(\Omega)$. It follows that $f - g \in \mathcal{N}_{\mathcal{M}}(\mathcal{O})$ as required by Theorem 21.9. ▲

If S is a set, then $\mathcal{P}_\omega(S)$ denotes the set of all finite subsets of S. In particular, $\mathcal{P}_\omega(\mathbb{N})$ denotes the set of the finite sets of natural numbers. The elements of the non-standard extension $^*\mathcal{P}_\omega(\mathbb{N})$ are called **hyperfinite sets**. They are, in general, infinite sets which are in one-to-one correspondence with sets of the form $\{1, 2, \ldots, \nu\}$ for some $\nu \in {}^*\mathbb{N}$. Let $\mathcal{P}_\omega(\mathbb{N})^\Omega$ be the set of the functions of the form $F : \Omega \to \mathcal{P}_\omega(\mathbb{N})$ and, similarly, let $^*\mathcal{P}_\omega(\mathbb{N})^{*\Omega}$ denote the set of the functions of the form $F : {}^*\Omega \to {}^*\mathcal{P}_\omega(\mathbb{N})$. For the non-standard extension $^*\left(\mathcal{P}_\omega(\mathbb{N})^\Omega\right)$ we have a strict inclusion $^*\left(\mathcal{P}_\omega(\mathbb{N})^\Omega\right) \subsetneqq {}^*\mathcal{P}_\omega(\mathbb{N})^{*\Omega}$. We say that $^*\left(\mathcal{P}_\omega(\mathbb{N})^\Omega\right)$ consists exactly of the *internal* functions in $^*\mathcal{P}_\omega(\mathbb{N})^{*\Omega}$. We should note that a fluent knowledge on internal hyperfinite functions is not necessary for the understanding of what follows.

We denote by $\mathcal{T}_d$ the usual topology on $\mathbb{R}^d$ and by $(\mathbb{R}^d, \mathcal{T}_d)$ the corresponding topological space.

23.3 Theorem. *The collection $\{\widehat{\mathcal{E}}_{\mathcal{M}}(\Omega)\}_{\Omega \in \mathcal{T}_d}$ is a* **sheaf of differential rings** *on $(\mathbb{R}^d, \mathcal{T}_d)$ under the restriction $\restriction$ in the sense that:*

(i) $(\forall \Omega \in \mathcal{T}_d)(\forall \widehat{f} \in \widehat{\mathcal{E}}_{\mathcal{M}}(\Omega))(\widehat{f} \restriction \Omega = \widehat{f})$.

(ii) $(\forall \Omega_1, \Omega_2, \Omega \in \mathcal{T}_d)(\forall \widehat{f} \in \widehat{\mathcal{E}}_{\mathcal{M}}(\Omega))(\Omega_1 \subseteq \Omega_2 \subseteq \Omega$ *implies* $(\widehat{f} \restriction \Omega_2) \restriction \Omega_1 = \widehat{f} \restriction \Omega_1$.

Let $\Omega = \bigcup_{\lambda \in \Lambda} \Omega_\lambda$ be an **open covering** *of $\Omega \in \mathcal{T}_d$ (for some index set Λ and some open sets $\Omega_\lambda \in \mathcal{T}_d$). Then:*

(iii) $(\forall \lambda \in \Lambda)(\widehat{f} \restriction \Omega_\lambda = 0)$ *implies* $\widehat{f} = 0$.

(iv) *Let $\{\widehat{f}_\lambda\}_{\lambda \in \Lambda}$, $\widehat{f}_\lambda \in \widehat{\mathcal{E}}_{\mathcal{M}}(\Omega_\lambda)$, be a* **coherent family** *of asymptotic functions in the sense that it satisfies the compatibility condition*

$$(\forall \lambda_1, \lambda_2 \in \Lambda) \left[\Omega_{\lambda_1} \cap \Omega_{\lambda_2} \neq \varnothing \Rightarrow \widehat{f}_{\lambda_1} \restriction (\Omega_{\lambda_1} \cap \Omega_{\lambda_2}) = \widehat{f}_{\lambda_2} \restriction (\Omega_{\lambda_1} \cap \Omega_{\lambda_2}) \right].$$

Then there exists $\widehat{f} \in \widehat{\mathcal{E}}_{\mathcal{M}}(\Omega)$ such that $\widehat{f} \restriction \Omega_\lambda = \widehat{f}_\lambda$ for all $\lambda \in \Lambda$.

(v) *The restriction $\restriction$ agrees with the* **differential ring operations** *in the sense that*

$$(\forall \Omega, \mathcal{O} \in \mathcal{T}_d)(\forall \widehat{f} \in \widehat{\mathcal{E}_{\mathcal{M}}}(\Omega))(\forall \alpha \in \mathbb{N}_0^d) \left(\mathcal{O} \subseteq \Omega \Rightarrow (\partial^\alpha \widehat{f}) \upharpoonright \mathcal{O} = \partial^\alpha (\widehat{f} \upharpoonright \mathcal{O}) \right).$$

$$(\forall \Omega, \mathcal{O} \in \mathcal{T}_d)(\forall \widehat{f}, \widehat{g} \in \widehat{\mathcal{E}_{\mathcal{M}}}(\Omega)) \left(\mathcal{O} \subseteq \Omega \Rightarrow (\widehat{f} + \widehat{g}) \upharpoonright \mathcal{O} = \widehat{f} \upharpoonright \mathcal{O} + \widehat{g} \upharpoonright \mathcal{O} \right).$$

$$(\forall \Omega, \mathcal{O} \in \mathcal{T}_d)(\forall \widehat{f}, \widehat{g} \in \widehat{\mathcal{E}_{\mathcal{M}}}(\Omega)) \left(\mathcal{O} \subseteq \Omega \Rightarrow (\widehat{f}\,\widehat{g}) \upharpoonright \mathcal{O} = (\widehat{f} \upharpoonright \mathcal{O})(\widehat{g} \upharpoonright \mathcal{O}) \right).$$

23.4 Remark (Sheaf Terminology). *A collection $\{\widehat{\mathcal{E}_{\mathcal{M}}}(\Omega)\}_{\Omega \in \mathcal{T}_d}$ which satisfies the properties (i) and (ii) is called* **presheaf** *on $(\mathbb{R}^d, \mathcal{T}_d)$. A presheaf $\{\widehat{\mathcal{E}_{\mathcal{M}}}(\Omega)\}_{\Omega \in \mathcal{T}_d}$ which satisfies (iii) and (iv) is called a* **sheaf** *on $(\mathbb{R}^d, \mathcal{T}_d)$. A sheaf is called a* **differential ring sheaf** *it it satisfies (v). For the relevant terminology we refer to A. Kaneko [39].*

Proof: (i) $\widehat{f} \upharpoonright \Omega = \widehat{f \,|\, {}^*\Omega} = \widehat{f}$ because ${}^*\Omega$ is the domain of f.

(ii) $(\widehat{f} \upharpoonright \Omega_2) \upharpoonright \Omega_1 = \widehat{(f \,|\, {}^*\Omega_2)} \upharpoonright \Omega_1 = \widehat{(f \,|\, {}^*\Omega_2) \,|\, {}^*\Omega_1} = \widehat{f \,|\, {}^*\Omega_1} = f \upharpoonright \Omega_1$ (as required) since ${}^*\Omega_1 \subseteq {}^*\Omega_2 \subseteq {}^*\Omega$ and Ω is the domain of f.

(iii) Suppose that $\lambda \in \Lambda$. We have $\widehat{f} \upharpoonright \Omega_\lambda = 0$ *iff* $\widehat{f \,|\, {}^*\Omega_\lambda} = 0$ *iff* $f \,|\, {}^*\Omega_\lambda \in \mathcal{N}(\Omega_\lambda)$ *iff* $(\forall x \in \mu(\Omega_\lambda))(f(x) \in \mathcal{M}_0)$ by Theorem 21.9. On the other hand $(\forall \lambda \in \Lambda)(\forall x \in \mu(\Omega_\lambda))(f(x) \in \mathcal{M}_0)$ *iff* $(\forall x \in \mu(\Omega))(f(x) \in \mathcal{M}_0)$ since $\bigcup_{\lambda \in \Lambda} \mu(\Omega_\lambda) = \mu(\Omega)$. Thus it follows $f \,|\, {}^*\Omega \in \mathcal{N}(\Omega)$ by Theorem 21.9 implying $\widehat{f} \upharpoonright \Omega = 0$ (as required).

(iv) Let $\Omega = \bigcup_{n=1}^\infty \mathcal{O}_n$ be a *locally finite countable covering* of Ω which is a *refinement* of $\Omega = \bigcup_{\lambda \in \Lambda} \Omega_\lambda$ in the sense that

(a) $\mathcal{O}_n \in \mathcal{T}_d$ and $\overline{\mathcal{O}}_n \subset\subset \Omega$.

(b) For every $K \subset\subset \Omega$ the set $\{n \in \mathbb{N} \mid K \cap \mathcal{O}_n \neq \varnothing\}$ is finite.

(c) There exists a sequence $\lambda \in \Lambda^{\mathbb{N}}$ such that $\overline{\mathcal{O}}_n \subset \Omega_{\lambda(n)}$ for all $n \in \mathbb{N}$.

Let $\{\varphi_n\}_{n \in \mathbb{N}}$ be a *smooth partition of unity* subordinate to $\{\mathcal{O}_n\}_{n \in \mathbb{N}}$ in the sense that:

(d) $\varphi_n \in \mathcal{D}(\mathcal{O}_n)$ for all $n \in \mathbb{N}$.

(e) $0 \leq \varphi_n(x) \leq 1$ for all $x \in \mathcal{O}_n$.

(f) $1 = \sum_{n=1}^\infty \varphi_n(x)$ for all $x \in \Omega$.

We recall that *every open covering has a locally finite countable covering refinement* and that *every locally finite countable covering has a smooth partition of unity* (A. Kaneko [39]). Notice that there exists $F \in \mathcal{P}_\omega(\mathbb{N})^\Omega$ such that

$$\sum_{n \in F(x)} \varphi_n(x) = 1,$$

for all $x \in \Omega$ and the function $f : \Omega \to \mathbb{C}$, defined by the formula

$$f(x) = \sum_{n \in F(x)} \varphi_n(x) \, f_{\lambda(n)}(x),$$

is in $\mathcal{E}(\Omega)$. Thus there exists $H \in {}^*\!\left(\mathcal{P}_\omega(\mathbb{N})^\Omega\right)$ such that

$$\sum_{n \in H(x)} {}^*\varphi_n(x) = 1,$$

for all $x \in {}^*\Omega$, by Transfer Principle (Theorem 4.8) which implies (trivially)

$$(33) \qquad f_\lambda(x) = f_\lambda(x) \sum_{n \in H(x)} {}^*\varphi_n(x),$$

for all $x \in {}^*\Omega$. We define the function $f : {}^*\Omega \to {}^*\mathbb{C}$ by the formula

$$(34) \qquad f(x) = \sum_{n \in H(x)} {}^*\varphi_n(x) \, f_{\lambda(n)}(x).$$

This is the function we are looking for. Indeed, we have $f \in {}^*\mathcal{E}(\Omega)$ by Transfer Principle because f is a hyperfinite sum (34) of functions in ${}^*\mathcal{E}(\Omega)$. Also $f \in \mathcal{M}_\mathcal{M}(\Omega)$ (by Transfer Principle again) because f is a hyperfinite sum (34) of functions in $\mathcal{M}_\mathcal{M}(\Omega)$. Suppose that $x \in \mu(\Omega_\lambda)$. After subtracting (33) from (34) we obtain:

$$f(x) - f_\lambda(x) = \sum_{n \in H(x)} {}^*\varphi_n(x) \left(f_{\lambda(n)}(x) - f_\lambda(x) \right).$$

This formula implies $f(x) - f_\lambda(x) \in \mathcal{M}_0$ since ${}^*\varphi_n(x)$ is a finite number and $f_{\lambda(n)}(x) - f_\lambda(x) \in \mathcal{M}_0$ by the compatibility condition. On the other hand, $f(x) - f_\lambda(x) \in \mathcal{M}_0$ implies $f - f_\lambda \in \mathcal{N}_\mathcal{M}(\Omega_\lambda)$ by Theorem 21.9. Thus $\widehat{f} \upharpoonright \Omega_\lambda = \widehat{f \mid {}^*\Omega_\lambda} = \widehat{f_\lambda \mid {}^*\Omega_\lambda} = \widehat{f_\lambda}$ (as required) by Lemma 23.2 since ${}^*\Omega_\lambda \subseteq {}^*\Omega$ and ${}^*\Omega_\lambda$ is the domain of f_λ.

(v) We have $(\partial^\alpha \widehat{f}) \restriction \mathcal{O} = \widehat{\partial^\alpha f} \restriction \mathcal{O} = \widehat{\partial^\alpha f \mid {}^*\mathcal{O}} = \partial^\alpha(\widehat{f \mid {}^*\mathcal{O}}) = \partial^\alpha(\widehat{f} \restriction \mathcal{O})$ as required. The verification of the sum and multiplication is similar and we leave it to the reader.

▲

The above result justifies the following definition.

23.5 Definition (Standard Support)**.** *Let Ω be two open sets of $\mathbb{R}^d$ and let $\widehat{f} \in \widehat{\mathcal{E}}_\mathcal{M}(\Omega)$. Let $\mathcal{O}$ be the maximal open subset of $\mathbb{R}^d$ such that $\widehat{f} \restriction \mathcal{O} = 0$ in $\widehat{\mathcal{E}}_\mathcal{M}(\mathcal{O})$. Then the set $\mathrm{supp}(\widehat{f}) = \mathbb{R}^d \setminus \mathcal{O}$ is called* **standard support** *(or simply* **support** *if no confusion can arise) of $\widehat{f}$.*

Te next result follows immediately from the above definition.

23.6 Proposition. *Every asymptotic function $\widehat{f} \in \widehat{\mathcal{E}_\mathcal{M}}(\Omega)$ has a (standard) support $\mathrm{supp}(\widehat{f})$ which is a* ***closed set of*** *Ω in the usual topology on $\mathbb{R}^d$.*

23.7 Theorem (Usual Support)**.** *The embedding $f \to {}^*\widehat{f}$ from $\mathcal{E}(\Omega)$ into $\widehat{\mathcal{E}_\mathcal{M}}(\Omega)$ is a sheaf homomorphism in the sense that $\widehat{{}^*(f \mid \mathcal{O})} = {}^*\widehat{f} \restriction \mathcal{O}$. Consequently, $\mathrm{supp}(f) = \mathrm{supp}({}^*\widehat{f})$, where $\mathrm{supp}(f)$ stands for the usual support of f in $\mathcal{E}(\Omega)$.*

Proof: We have ${}^*f \mid {}^*\mathcal{O} = {}^*(f \mid \mathcal{O})$ by Transfer Principle (Theorem 4.8). Thus $\widehat{{}^*(f \mid \mathcal{O})} = \widehat{{}^*f \mid {}^*\mathcal{O}} = {}^*\widehat{f} \restriction \mathcal{O}$ as required.

24   A Canonical form of the Algebras $\widehat{\mathcal{E}}_\mathcal{M}(\Omega)$

So far we constructed the algebra $\widehat{\mathcal{E}}_\mathcal{M}(\Omega)$ of asymptotic functions by the **following scheme:**

1. We choose a convex subring $\mathcal{M}$ of ${}^*\mathbb{C}$ (Section **??**).

2. We construct the ideal $\mathcal{M}_0$ and the algebraically closed field $\widehat{\mathcal{M}}$ (Section **??**). Notice that $\widehat{\mathcal{M}}$ can be embedded as a subfield of ${}^*\mathbb{C}$ (which is important for what follows) and the image of $\widehat{\mathcal{M}}$ into ${}^*\mathbb{C}$ under this embedding is a maximal field $\mathbb{M} \in \mathcal{M}ax(\mathcal{M})$ (Definition 14.6).

3. We define $\mathcal{M}_\mathcal{M}(\Omega)$ and $\mathcal{N}_\mathcal{M}(\Omega)$ and the algebra $\widehat{\mathcal{E}}_\mathcal{M}(\Omega) = \mathcal{M}_\mathcal{M}(\Omega)/\mathcal{N}_\mathcal{M}(\Omega)$ over the field of scalars $\widehat{\mathcal{M}}$ (Section **??**)

We shall presented an **alternative construction** of $\widehat{\mathcal{E}}_{\mathcal{M}}(\Omega)$: We start from a given (already chosen or constructed) an algebraically closed subfield $\mathbb{M}$ of $^*\mathbb{C}$ and then we define the algebra $\widehat{\mathcal{E}}_{\mathcal{M}}(\Omega)$ directly from $\mathbb{M}$. The connection between the two construction is given by the formula:

$$\mathcal{M} = \{z \in {}^*\mathbb{C} \mid (\exists \zeta \in \mathbb{M})(|z| \leq |\zeta|)\}.$$

24.1 Definition (Asymptotic Functions Generated by a Field). *Let $\mathbb{M}$ be an algebraically closed subfield of $^*\mathbb{C}$. We let*

$$\mathbb{M}(\Omega) = \{f \in {}^*\mathcal{E}(\Omega) \mid (\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega)(\partial^\alpha f(x) \in \mathbb{M})\},$$
$$\mathbb{M}_0(\Omega) = \{f \in {}^*\mathcal{E}(\Omega) \mid (\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega)(\partial^\alpha f(x) = 0)\},$$

*and let $\widehat{\mathbb{M}}(\Omega) = \mathbb{M}(\Omega)/\mathbb{M}_0(\Omega)$ be the corresponding factor ring. We say that $\widehat{\mathbb{M}}(\Omega)$ **is generated by the field** $\mathbb{M}$.*

24.2 Theorem. *Let $\mathbb{M}$ be an algebraically closed subfield of $^*\mathbb{C}$. Then $\mathbb{M}(\Omega)$ is a differential ring and $\mathbb{M}_0(\Omega)$ is a differential ideal in $\mathbb{M}(\Omega)$ and we also have*

$$\mathbb{M}_0(\Omega) = \{f \in \mathbb{M}(\Omega) \mid (\forall x \in \mu(\Omega)(f(x) = 0)\}.$$

Consequently, $\widehat{\mathbb{M}}(\Omega)$ is both a differential ring and a differential algebra over the field $\mathbb{M}$.

Proof: The statement about $\mathbb{M}(\Omega)$, $\mathbb{M}_0(\Omega)$ and $\widehat{\mathbb{M}}(\Omega)$ follows directly from the above definition. The proof of the formula for $\mathbb{M}_0(\Omega)$ is almost identical to the proof of Theorem 21.9 and leave it to the reader. ▲

24.3 Theorem (An Isomorphism). *Let $\mathcal{M}$ be a convex subring of $^*\mathbb{C}$ and $\mathbb{M} \in \mathcal{M}ax(\mathcal{M})$ be a maximal field within $\mathcal{M}$ (Definition 14.6). Then $\widehat{\mathcal{E}_{\mathcal{M}}}(\Omega)$ and $\widehat{\mathbb{M}}(\Omega)$ are isomorphic differential algebras over the field $\mathbb{M}$ under pointwise characterization of $\widehat{\mathcal{E}_{\mathcal{M}}}(\Omega)$:*

$$\widehat{f} \in \widehat{\mathcal{E}_{\mathcal{M}}}(\Omega) \to f \in \widehat{\mathcal{M}}^{\mu_{\mathcal{M}}(\Omega)},$$

(Section 22).

Proof: We have $\widehat{\mathcal{M}} = \widehat{\mathbb{M}}$ by part (i) of Theorem 14.8 and also we have

(35) $\qquad \mu_{\mathcal{M}}(\Omega) = \{r + dx \mid r \in \Omega, \, dx \in \Re(\widehat{\mathbb{M}}^d), \, ||dx|| \approx 0\}.$

(compare with (30) in Section 22). Thus $\widehat{\mathcal{M}}^{\mu_{\mathcal{M}}(\Omega)} = \widehat{\mathbb{M}}^{\mu_{\mathbb{M}}(\Omega)}$. On the other hand $\widehat{\mathbb{M}}$ and $\mathbb{M}$ are field isomorphic by part (ii) of Theorem 14.8. Thus $\widehat{\mathbb{M}}^{\mu_{\mathbb{M}}(\Omega)}$ and $\mathbb{M}^{\mu_{\mathbb{M}}(\Omega)}$ are ring isomorphic. The theorem is complete. ▲

24.4 Corollary. *Let $\mathcal{M}$ be a convex subring of $^*\mathbb{C}$ and $\mathbb{M}_1$, $\mathbb{M}_2 \in \mathcal{M}ax(\mathcal{M})$ be two maximal fields. Then $\widehat{\mathbb{M}_1}(\Omega)$ and $\widehat{\mathbb{M}_2}(\Omega)$ are isomorphic differential algebras over the field $\widehat{\mathcal{M}}$.*

24.5 Remark (A Canonical Form). *Based on the above results we shall sometimes identify notationally the algebras of asymptotic functions $\widehat{\mathcal{E}_{\mathcal{M}}}(\Omega)$ and $\widehat{\mathbb{M}}(\Omega)$ writing simply*

$$\widehat{\mathcal{E}_{\mathcal{M}}}(\Omega) = \widehat{\mathbb{M}}(\Omega).$$

*We say that $\widehat{\mathbb{M}}(\Omega)$ is a **canonical form of the algebra** $\widehat{\mathcal{E}_{\mathcal{M}}}(\Omega)$. We should mention that the theory of $\widehat{\mathbb{M}}(\Omega)$ is somewhat simpler and more elegant than the theory of $\widehat{\mathcal{E}_{\mathcal{M}}}(\Omega)$. However, $\widehat{\mathcal{E}_{\mathcal{M}}}(\Omega)$ is more easily supported by examples because one can more easily produce examples of convex subrings $\mathcal{M}$ of $^*\mathbb{C}$ rather then to produce examples of algebraically closed subfileds $\mathbb{M}$ of $^*\mathbb{C}$ (see the end of Section ??).*

24.6 Example (Levi-Civita Field). *Let ρ be a positive infinitesimal in $^*\mathbb{R}_+$ and let $\mathbb{M} = \mathbb{C}\langle\rho\rangle$ denote the field of the T. Levi-Civita [54] power series with complex coefficients, i.e. series of the form*

$$\sum_{n=0}^{\infty} a_n \, \rho^{r_n},$$

where (a_n) is a sequence in $\mathbb{C}$ and (r_n) is a strictly increasing sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} a_n = \infty$ (we shall abbreviate all these as $r_0 < r_1 < r_2 < \cdots \to \infty$). We should mention that $\mathbb{C}\langle\rho\rangle$ is an algebraically closed field (as required in the above definition). Also the Levi-Civita series are convergent in $^*\mathbb{C}$ under the valuation norm $||\cdot||_\rho$. Let $\widehat{\mathbb{M}}(\Omega) = \widehat{\mathbb{C}\langle\rho\rangle}(\Omega)$ be the algebra of generalized functions generated by the field $\mathbb{C}\langle\rho\rangle$. Then every $\widehat{f} \in \widehat{\mathbb{C}\langle\rho\rangle}(\Omega)$ can be presented in the form

$$\widehat{f}(x) = \sum_{n=0}^{\infty} \widehat{a_n}(x) \, \rho^{r_n}$$

for every $x \in \mu_{\mathcal{M}}(\Omega)$, where $\widehat{a_n} : \mu_{\mathcal{M}}(\Omega) \to \mathbb{C}$.

25 Convolution in Non-Standard Setting

25.1 Definition (Convolution). **(i)** Let $T \in \mathcal{D}'(\Omega)$ and let $T : \mathcal{D}(\Omega) \to \mathbb{C}$ be the corresponding mapping. We define the **non-standard extension** $^*T : {}^*\mathcal{D}(\Omega) \to {}^*\mathbb{C}$ of T by the formula

$$\langle {}^*T, \langle \varphi_i \rangle \rangle = \langle \langle T, \varphi_i \rangle \rangle,$$

where $\langle \varphi_i \rangle \in {}^*\mathcal{D}(\Omega)$.

(ii) Let $T \in \mathcal{E}'(\Omega)$ and $\langle D_i \rangle \in {}^*\mathcal{D}(\mathbb{R}^d)$. We define the **convolution** between $^*T$ and $\langle D_i \rangle$ by the formula

$$^*T \star \langle D_i \rangle = \langle T \star D_i \rangle,$$

where $T \star D_i$ is the usual convolution between T and D_i in the sense of distribution theory (i.e. $\langle T(\xi), D_i(x - \xi) \rangle$ for every $x \in \Omega$ and every $i \in \mathcal{I}$).

25.2 Lemma. *For every $T \in \mathcal{E}'(\Omega)$ and every $D \in {}^*\mathcal{D}(\mathbb{R}^d)$ we have $^*T \star D \in {}^*\mathcal{E}(\Omega)$.*

26 Schwartz Distributions in $^\rho\mathcal{E}(\Omega)$

If $f \in \mathcal{L}^1_{loc}(\Omega)$, we denote by $T_f \in \mathcal{D}'(\Omega)$ the Schwartz distribution with kernel f, i.e.

$$\langle T_f, \varphi \rangle = \int_\Omega f(x)\varphi(x)\,dx,$$

for all $\varphi \in \mathcal{D}(\Omega)$. Recall that $\mathcal{E}(\Omega)$ is a differential subring of $^\rho\mathcal{E}(\Omega)$ under the embedding

$$\mathcal{E}(\Omega) \hookrightarrow {}^\rho\mathcal{E}(\Omega),$$

defined by the mapping $f \to \widehat{{}^*f}$, where $^*f$ is the non-standard extension of f (i.e. $^*f = \langle f_i \rangle$, $f_i = f$ for all $i \in \mathcal{I}$) and $\widehat{{}^*f}$ stands for the corresponding equivalence class (see the beginning of Section **??**).

26.1 Theorem (Existence of an Embedding). *There exists an embedding $\Sigma_\Omega : \mathcal{D}'(\Omega) \to {}^\rho\mathcal{E}(\Omega)$ which preserves the sheaf-properties and the linear operations in $\mathcal{D}'(\Omega)$ (including partial differentiation) and such that $\Sigma_\Omega(T_f) =$*

$\Sigma_\Omega(*f)$ *for every* $f \in \mathcal{E}(\Omega)$. *Consequently, the multiplication in* $^\rho\mathcal{E}(\Omega)$ *re-duces to the usual pointwise multiplication on* $\mathcal{E}(\Omega)$. *We summarize this in:*

$$\mathcal{E}(\Omega) \hookrightarrow \mathcal{D}'(\Omega) \hookrightarrow {}^\rho\mathcal{E}(\Omega)$$

Proof: We shall separate the proof in numerous definitions and lemmas:

26.2 Definition (ρ-Delta Function). $D \in {}^*\mathcal{E}(\mathbb{R}^d)$ is called a ρ-**delta func-tion** if:

1. $||x|| \not\approx 0$ implies $D(x) = 0$. (**Lemma:** There exists a **positive in-finitesimal**, say ρ, such that $||x|| \leq \rho$ implies $D(x) = 0$).

   The next conditions on D depend on the choice of ρ:

2. $\int_{||x|| \leq \rho} D(x)\, dx - 1 \in \mathcal{N}_\rho(^*\mathbb{C})$.

3. $\int_{||x|| \leq \rho} D(x)\, x^\alpha\, dx \in \mathcal{N}_\rho(^*\mathbb{C})$ for all $|\alpha| \neq 0$.

4. $D \in \mathcal{M}_\rho(^*\mathcal{E}(\mathbb{R}^d))$, i.e.

$$(\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\mathbb{R}^d))\left(\partial^\alpha D(x) \in \mathcal{M}_\rho(^*\mathbb{C})\right).$$

26.3 Theorem. *There exists a ρ-delta function D.*

Proof: : For the original proof we refer to (M. Oberguggenberger and T. Todorov [68]). Here is a **summary of this result**:
  Step 1) For every $n \in \mathbb{N}$, we define the set of test-functions:

(36) $\mathcal{B}_n = \{\varphi \in \mathcal{D}(\mathbb{R}^d) :$

$$\int_{\mathbb{R}^d} \varphi(x)\, dx = 1,$$

$$\int_{\mathbb{R}^d} x^\alpha \varphi(x)\, dx = 0 \ \text{ for all } \ \alpha \in \mathbb{N}_0^d,\ 1 \leq |\alpha| \leq n,$$

$$||x|| \geq 1/n \Rightarrow \varphi(x) = 0,$$

$$1 \leq \int_{\mathbb{R}^d} |\varphi(x)|\, dx < 1 + \frac{1}{n} \}.$$

26.4 Lemma (Properties of $\mathcal{B}_n$). (B_1) $\mathcal{B}_n \neq \varnothing$ *for all* n.
   (B_2) $\mathcal{D}(\mathbb{R}^d) = \mathcal{B}_0 \supset \mathcal{B}_1 \supset \mathcal{B}_2 \supset \mathcal{B}_3 \supset \dots$. *(Thus* $\mathcal{B}_n \cap \mathcal{B}_n = \mathcal{B}_{\max(m,n)}$*).*
   (B_3) $\cap_n \mathcal{B}_n = \varnothing$.

Step 2) Find the **non-standard extension** of $\mathcal{B}_n$:

(37) $^*\mathcal{B}_n = \{\varphi \in {}^*\mathcal{D}(\mathbb{R}^d) :$

$$\int_{^*\mathbb{R}^d} \varphi(x)\, dx = 1,$$

$$\int_{^*\mathbb{R}^d} x^\alpha \varphi(x)\, dx = 0 \ \text{ for all } \ \alpha \in \mathbb{N}_0^d, \ 1 \leq |\alpha| \leq n,$$

$$\|x\| \geq 1/n \Rightarrow \varphi(x) = 0,$$

$$1 \leq \int_{^*\mathbb{R}^d} |\varphi(x)|\, dx < 1 + \frac{1}{n} \}.$$

Step 3) Let M be an infinitely large positive number in $\mathcal{M}_\rho({}^*\mathbb{R})$. For example, $M = |\ln \rho|$ will do. Define the internal sets:

$$\mathcal{A}_n = \{\varphi \in {}^*\mathcal{B}_n : \quad {}^*\sup_{\|x\| \leq 1/n} |\partial^\alpha \varphi(x)| < \frac{M}{n} \ \text{ for all } \ |\alpha| \leq n\},$$

We observe that (trivially) $^*\mathcal{D}(\mathbb{R}^d) \supset \mathcal{A}_1 \supset \mathcal{A}_2 \supset \dots$. Also, $\mathcal{A}_n \neq \varnothing$ for all n. Indeed, $\varphi \in \mathcal{B}_n$ implies $^*\varphi \in \mathcal{A}_n$ since

$$^*\sup_{\|x\| \leq 1/n} |\partial^\alpha ({}^*\varphi(x))| = \sup_{\|x\| \leq 1/n} |\partial^\alpha \varphi(x)| < \frac{M}{n},$$

and $\sup_{\|x\| \leq 1/n} |\partial^\alpha \varphi(x)|$ is a real number and M/n is an infinitely large positive number for any $n \in \mathbb{N}$. Thus there exists

$$\Theta \in \bigcap_{n=1}^\infty \mathcal{A}_n \neq \varnothing,$$

by Saturation Principle (Theorem 4.5). Notice that Θ **satisfies all properties (1)-(4)** of the definition of ρ-delta function **except (possibly) the property (5)**.

Step 3) The non-standard function $D \in {}^*\mathcal{D}(\mathbb{R}^d)$, defined by the formula

$$D(x) = \rho^{-d} \Theta(x/\rho),$$

is the ρ-delta function we are looking for.

26.5 Definition. The mapping $T \to Q_\Omega \left({}^*T \star D\right)$ from $\mathcal{E}'(\Omega)$ to $^\rho\mathcal{E}(\Omega)$ is the embedding of the space of distributions with compact support in Ω.

Step 4)

26.6 Definition (ρ-Cut-Off Function). $\Pi_\Omega \in {}^*\mathcal{D}(\Omega)$ is called a ρ-**cut-off function** for the open set $\Omega \subseteq \mathbb{R}^d$ if

(a) $\Pi_\Omega(x) = 0$ for all $x \in \mu(\Omega)$.

(b) $\mathrm{supp}(\Pi_\Omega) \subseteq \{x \in {}^*\Omega \mid {}^*d(x, \partial\Omega) \geq \rho\}$

26.7 Lemma. *There exists a ρ-cut-off-function.*

Proof: Let $\Omega_\rho = \{x \in {}^*\Omega \mid {}^*d(x, \partial\Omega) \geq 2\rho, ||x|| < 1/\rho\}$ and let χ be the characteristic function of Ω_ρ. The function $\Pi_\Omega = \chi \star D$ is the ρ-cut-off function we are looking for. $\blacktriangle$

26.8 Definition. The mapping $T \to Q_\Omega \left({}^*T\Pi_\Omega\right) \star D$ from $\mathcal{D}'(\Omega)$ to ${}^\rho\mathcal{E}(\Omega)$ is the embedding the existence of which was stated in Theorem 26.1.

The proof of Theorem 26.1 is complete. $\blacktriangle$

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